

Adaptation, Comparison and Practical Implementation of Fairness Schemes in Kidney Exchange Programs

William St-Arnaud^{a,b}, Margarida Carvalho^{a,b}, Golnoosh Farnadi^{c,b}

^a*Département d'Informatique et de Recherche Opérationnelle & CIRRELT, Université de Montréal, 2920, chemin de la Tour, Montréal, Québec H3T 1J4, Canada*

^b*Mila-Quebec AI Institute, 6666 St-Urbain, Montréal, Québec, H2S 3H1, Canada*

^c*School of Computer Science, McGill University, 3480 Rue University, Montréal, H3A 2A7, Canada*

Abstract

Kidney Exchange Programs (KEPs) typically maximize overall patient benefit through donor exchanges. This aggregation of benefits (utilitarian objective) calls into question potential individual patient disparities in terms of access to transplantation in KEPs. Moreover, current KEP policies are all-or-nothing, meaning that only one exchange plan is determined - each patient is either selected or not as part of that unique solution. In this work, we extend the space of policies by seeking a lottery over the set of exchange plans that contemplates the (ex-ante) probability of patients being in a solution. To guide the determination of our policy, we adapt popular fairness schemes to KEPs to balance the usual approach of maximizing the utilitarian objective. Different combinations of fairness and utilitarian objectives are modelled as conic programs with an exponential number of variables. We propose a column generation approach to solve them effectively in practice. Finally, we make an extensive comparison of the different schemes in terms of the balance of utility and fairness score, and validate the scalability of our methodology for benchmark instances from the literature.

Keywords: Combinatorial optimization, Kidney exchange programs, Fairness, Conic programming, Lottery

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1. Introduction

Kidney Exchange Programs (KEPs) allow patients that have an incompatible donor to participate in exchange cycles and chains, thereby providing an additional alternative in the search for an organ. KEPs are implemented in several countries, *e.g.*, South Korea [51], the Netherlands [25], the United Kingdom [47] and Canada [46].

The purpose of KEPs is to find a solution, namely an exchange plan, that consists of a set of donor exchanges resulting in compatible transplants for the patients. Often, the solution selection is guided by the objective of maximizing the number of transplants. This objective may result in some patients being excluded from that optimal exchange plan. To address this issue, several approaches to fairness in KEPs have been explored. These include maximizing the number of hard-to-match patients in a selected exchange plan [32] and balancing participation from different countries involved in a KEP pool [11, 43, 9]. These methods modify the objective so that the unit value associated with each

transplant is replaced by a weight, or add constraints to ensure that a minimum fairness metric value is achieved. This results in a combination of both fairness aspects and overall patient welfare.

These approaches to fairness form the basis of deterministic policies for KEPs. Regardless of the choice of the objective to be optimized and the considered constraints by a deterministic policy, only the ex-post *value* of an exchange plan is taken into account. As we will see in Section 4, some exchange plans under deterministic policies are ex-ante guaranteed to not be selected, even if they are optimal. For this reason, it becomes relevant to consider lottery (random) policies for KEPs. They offer a generalization of deterministic policies, while also enabling decision-makers to guarantee ex-ante probabilities of certain outcomes to various patients.¹

The use of lottery policies motivates the need for a detailed study of fairness in KEPs to guide the assignment of probabilities to exchange plans. We formalize our analysis under the lens of fairness schemes [10] and we adapt (*i.e.*, model) popular fairness scores to the context of KEP: individual fairness [33], Nash’s standard of comparison [49, 10], Aristotle’s principle of fairness [2], and Rawlsian justice [53]. Moreover, we model fairness schemes in which the lotteries’ computation is driven by a single-objective (a fairness score) and a combination of two objectives (an utilitarian and a fairness score), the latter using the Social Welfare Program [36] and the Nash Social Welfare Program [23]. Since for each fairness scheme we build a lottery, our decision variables correspond to the probability of selecting an exchange plan. Thus, we first show that all our fairness schemes can be modeled as conic programs with an exponential number of variables, and second, we devise a column generation to solve these programs efficiently. As part of our experiments, we explore the quality of solutions returned by each mathematical program and fairness scheme combination in terms of a utilitarian (*i.e.*, number of transplants) and fairness objectives, as well as the effectiveness of our column generation methodology for their applicability to synthetic KEP instances of realistic size.

Our contributions are the following: 1. The interpretation and modeling of popular fairness concepts (scores) in the context of KEPs, which allows a unified view of fairness, including the broadly-used group fairness; 2. The construction of fairness schemes through the determination of lottery policies for KEPs based on a single-fairness score, and on the combination of utilitarian and fairness scores (bi-objective); 3. For the bi-objective schemes, the use of the Social Welfare Program and the Nash Social Welfare Program to find a Pareto-optimal lottery policy; 4. The formulation of our fairness schemes as conic programs and the design of a column-generation approach to solve them efficiently; 5. An analysis of the duals of our bi-objective approaches to interpret the weights in the subproblems during column generation and a new definition of *hard-to-match* patients based on structural constraints of the KEP graph instead of common approaches based on *clinical compatibility*; 6. A detailed experimental study comparing our fairness schemes, and providing insights to policy-makers on their benefits and limitations.

2. Related work

In the context of KEPs, the notion of *group fairness* has been explored. It can take the shape of ensuring fairness towards pre-determined groups of patients, *e.g.*, those that are deemed hard-to-match (*i.e.*, highly-sensitized). However, this may negatively affect the total number of transplants realized

¹Lotteries define a probability distribution over exchange plans, hence the generalization of deterministic policies.

under an exchange plan [31]. In this setting, the definition of the *price of fairness* provides a measure of the cost of prioritizing these pre-determined groups to the total number of transplants [10, 31]. Dickerson et al. [31] show that the price of fairness can become quite high when maximizing the number of highly-sensitized patients. They introduce a threshold to balance these two objectives: the threshold corresponds to the percentage of highly-sensitized patients that are to be included in the solution; it is given as a constraint and chosen empirically. Instead, Freedman et al. [35] rely on maximizing the number of transplants and breaking ties using the number of highly-sensitized patients. Riascos-Álvarez et al. [54] propose a decomposition method to contend with long cycles and chains in large instances. They achieve higher matching efficiency, especially for highly-sensitized patients, by prioritizing chains over cycles. Meanwhile, Glorie et al. [40] propose an iterative branch-and-price algorithm to help solve hierarchical and weighted multi-objective KEPs. They manage to accommodate long chains on instances of realistic size. The aforementioned fairness approaches in KEPs are not exhaustive. For examples related to game theory, Sönmez and Ünver [61], Ashlagi and Roth [6], Carvalho and Lodi [22] and Klimentova et al. [43] all cast the fairness component of KEPs in a framework where agents (representing hospitals, regions or countries) aim to maximize the benefit for their own patients; roughly, the goal is to find a policy encouraging agents to participate in a joint program. In this work, we do not focus on multi-agent KEPs and hence, we analyze fairness exclusively from the patients’ point of view.

A natural approach to resource allocation involves the use of lotteries, such as in the case of two-sided matching markets under preferences. These include applications like school choice [7, 27] and resident allocation [16]. Lotteries can help break ties when multiple possible allocations exist and can be restricted to a particular set of desirable outcomes, as is the case in Israel’s resident allocation program [16]. Relevant within this context is literature related to *procedural fairness*. Fairness in resource allocation problems is often concerned with the properties of a particular allocation in order to assess its desirability. An alternative to fairness of outcome (*i.e.*, ex-post fairness) is *procedural fairness*. This notion places an emphasis on the procedure that is deployed to select a particular allocation. Procedural fairness also presents a framework through which we can analyze distributions over allocations by examining the probabilities of various outcomes beforehand (*i.e.*, ex-ante fairness). Bolton et al. [15] sets out to compare fair outcomes with procedural fairness in the context of games. In various experiments, the authors show that participants were indifferent when choosing their preferred mode of play. A fair procedure was deemed satisfying to the participants even if the realized outcome did not favour them. The authors therefore argue that a fair procedure can be used as a substitute for a fair outcome. Tyler [62] also provides supporting evidence for this fact by demonstrating that a fair procedure is the determinant factor in the analysis of the fairness of an outcome. The study also shows that individuals’ judgements about the fairness of a procedure is not influenced by the outcome if it was judged to be fair beforehand. In the context of KEPs, the ex-ante analysis for lottery policies (randomized mechanisms) has been covered in Ashlagi et al. [4], Caragiannis et al. [21], Ashlagi and Roth [6] to guarantee that hospitals have incentive to participate in a joint KEP due to their (ex-ante) expected individual utility; so even if the realized outcome is not *ideal* for some individuals, the ex-ante fairness makes the policy acceptable. In what follows, whenever we discuss “expectation” or “expected value”, we implicitly refer to the ex-ante value guaranteed by a distribution over exchange plans (*i.e.*, a lottery). Following this line of research on fair procedures, we highlight the *individual fairness* approach found in Farnadi et al. [33], although it is restricted to lotteries over

optimal exchange plans, *i.e.*, exchange plans achieving the maximum number of transplants, and is only applicable to small instances. Their goal is to balance individual patient probabilities of being selected in an exchange plan. By definition, individual fairness entails a null *price of fairness* to the detriment of maybe excluding some patients from consideration, *i.e.*, those that cannot be part of an optimal exchange plan. However, one can also consider an ex-ante fair procedure that consists of probability distributions over exchange plans that are not necessarily optimal in the (expected) number of transplants. This is what we set out to do in this article. Extending the support for our lotteries can allow us to give a non-zero probability of being in an exchange plan to patients who are not in optimal solutions. Multiple fairness concepts exist outside of individual fairness and we employ some of the most commonly discussed in our analysis. In addition, we devise a methodology to apply them to real-sized instances. As a final remark on related literature, we highlight the work of Demeulemeester et al. [28] who, building on an earlier working version of the present manuscript, extended the use of fairness through lotteries to general integer programming problems.

3. Kidney Exchange Programs

A KEP consists of a set of incompatible patient-donor pairs and a set of non-directed donors (NDDs), altruistic donors registered without a patient. Each incompatible patient-donor pair contains a patient necessitating a kidney and a (incompatible) donor willing to donate a kidney. This incompatibility can, for example, be due to bloodtype or tissue type mismatch [18]. The program seeks to find compatible donor exchanges between its pairs and also with the NDDs. The objective is to form an exchange plan, in such a way that every patient relinquishing their donor also receives a donor in turn. This can be thought of as a form of bartering involving multiple parties.

Formally, a KEP instance can be represented as a graph $G = (V, A)$, where $V = P \cup N$ consists of the set of incompatible patient-donor pairs P and the set of non-directed donors N , and where A is the set of arcs between these vertices. We have $(i, j) \in A$ if and only if the donor of vertex i is compatible with the patient from vertex j . One can realize that no $(i, j) \in A$ exists for $j \in N$, *i.e.*, no NDD can have an arc pointing to them. For an illustration, we refer the reader to Figure 1. Vertices v_1, v_2, v_3, v_4, v_5 and v_6 represent incompatible patient-donor pairs. They can swap donors to match their respective patients with a compatible organ. Vertex v_7 (shaded in gray) represents an NDD.

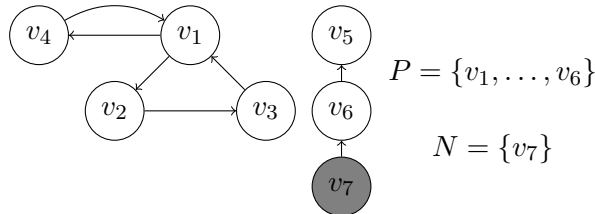


Figure 1: Example of a KEP graph

Exchange plans consist of vertex-disjoint *cycles* and *chains*. A cycle is formed with incompatible patient-donor pairs, while a chain is a (simple) path starting in an NDD. The *length* of a cycle (and chain) is the number of transplants in it (*i.e.*, one transplant for each vertex from P); it also corresponds to the number of transplants given by selecting this chain or cycle. In Figure 1, a feasible exchange plan can be given by the cycle (v_1, v_2, v_3, v_1) of length 3 and the chain (v_7, v_6, v_5) of length 2. An alternative exchange plan could instead involve the cycle (v_4, v_1, v_4) and the same chain. For ethical

reasons, the transplant operations corresponding to a selected cycle must be performed simultaneously. Therefore, due to transplant logistics, most KEPs impose a limit K on the length of the cycles, and often, also a limit K' on the length of the chains.

The typical goal of a KEP is the determination of an exchange plan maximizing the benefit of the patients. To this end, many integer programming formulations have been proposed as well as decomposition techniques to tackle them [1, 57, 29, 54, 24, 3]. We describe the *hybrid position-indexed edge formulation* [29] since it will be used within our decomposition methodology. We focus on this formulation as it is simple, non-exponential in size, can be used directly with modern solvers and has been shown to solve realistic-sized instances. However, we note that any other formulation can be easily adopted in our proposed methodology. To simplify the exposition, we assume that $K' = K$. Here, for each $l \in V$, we create a copy $G^l = (V^l, A^l)$ of the graph G . The sets V^l contain all vertices with indices $i \geq l$ and the sets A^l are the subsets of arcs induced by each V^l . We also define the following sets for $((i, j), l) \in \coprod_{l \in V} A^l$ and $(i, j) \in A$ respectively²:

$$\mathcal{K}(i, j, l) = \begin{cases} \{1\} & i = l \\ \{2, \dots, K-1\} & i, j > l \\ \{2, \dots, K\} & j = l. \end{cases} \quad \mathcal{K}'(i, j) = \begin{cases} \{1\} & i \in N \\ \{2, \dots, K\} & i \in P. \end{cases}$$

The purpose of having many graph copies is to eliminate possible symmetries when expressing cycles. For graph copy l , we only have vertices with indices greater or equal to l . This amounts to selecting a unique starting point in a cycle (which will be vertex l for the l -th copy). Because of the definition of $\mathcal{K}(i, j, l)$, we can recover the valid positions of each arc in a cycle contained in graph copy l (and by extension, the position of each vertex). The sets $\mathcal{K}'(i, j)$ serve the same function but correspond to chains. Because of the fact that chains have start and terminal vertices, they can be described independently of the graph copies, hence the notation of $\mathcal{K}'(i, j)$ without l as an argument.

We define the binary variables x_{ijk}^l which take value 1 if the donation of a kidney from vertex i to vertex j appears in the k -th position of a cycle in the l -th graph copy. We also define the binary variables y_{ijk} which take value 1 if the donation of a kidney from vertex i to vertex j appears in the

² $\coprod$ is the disjoint union operator.

k -th position of a chain. The formulation is as follows:

$$\begin{aligned}
& \max \quad \sum_{l \in P} \sum_{(i,j) \in A^l} \sum_{k \in \mathcal{K}(i,j,l)} w_{ij} x_{ijk}^l + \sum_{(i,j) \in A} \sum_{k \in \mathcal{K}'(i,j)} w_{ij} y_{ijk} \quad (\text{HPIEF}) \\
(c_1) \text{ s.t.} \quad & \sum_{l \in P} \sum_{j: (j,i) \in A^l} \sum_{k \in \mathcal{K}(j,i,l)} x_{jik}^l + \sum_{j: (j,i) \in A^l} \sum_{k \in \mathcal{K}'(j,i)} y_{jik} \leq 1 \quad \forall i \in P \\
(c_2) \quad & \sum_{\substack{j: (j,i) \in A^l \wedge \\ k \in \mathcal{K}(j,i,l)}} x_{jik}^l = \sum_{\substack{j: (i,j) \in A^l \wedge \\ k+1 \in \mathcal{K}(i,j,l)}} x_{ij(k+1)}^l \quad \begin{cases} \forall l \in V \\ \forall i \in \{l+1, \dots, |V|\} \\ \forall k \in \{1, \dots, K-1\} \end{cases} \\
(c_3) \quad & \sum_{j: (i,j) \in A} y_{ij1} \leq 1 \quad \forall i \in N \\
(c_4) \quad & \sum_{\substack{j: (j,i) \in A \wedge \\ k \in \mathcal{K}'(j,i)}} y_{jik} \geq \sum_{j: (i,j) \in A} y_{ij(k+1)} \quad \begin{cases} \forall i \in P \\ \forall k \in \{1, \dots, K-1\} \end{cases} \\
& x_{ijk}^l \in \{0, 1\} \quad \begin{cases} \forall l \in V \\ \forall (i,j) \in A^l \\ \forall k \in \mathcal{K}(i,j,l) \end{cases} \\
& y_{ijk} \in \{0, 1\} \quad \begin{cases} (i,j) \in A \\ k \in \mathcal{K}'(i,j) \end{cases}
\end{aligned}$$

where w_{ij} is the weight associated with the benefit to the patient in vertex j when receiving a transplant from the donor in vertex i . Constraint c_1 in (HPIEF) ensures that no pair is part of more than one cycle or chain. Constraint c_2 (flow conservation) ensures that whichever pair gives a kidney also receives one in return. Constraint c_3 verifies that each NDD (*i.e.*, $i \in N$) gives to at most one patient. Constraint c_4 allows a chain of current length $L < K$ to be extended by one at position $L + 1$. In other words, if vertex i gives to vertex j at position $k + 1$, vertex i must necessarily have received an organ from some vertex at position k in a chain. As a remark, since most KEPs' primary goal is to maximize the number of transplants, we generally have w_{ij} equal to 1. Non-unitary weights may correspond to utility values when receiving a kidney from a particular donor.

4. Lottery and fair procedure: the role of distributions

With HPIEF, a deterministic exchange is determined. Throughout this article, we take a broader perspective and we use probability distributions over exchange plans, *i.e.*, lotteries. Lottery policies allow for the generalization of deterministic policies by making it possible to assign a probability distribution to a subset of (deterministic) exchange plans (*i.e.*, solutions).

While it is possible to account for the ex-post fairness of solutions in a deterministic setting, fair lotteries are essentially fair procedures that can also give consideration to the ex-ante fairness of a KEP policy. While Bolton et al. [15] show that fair lotteries are equivalent to fair outcomes, we also remark that the two notions of fairness (*i.e.* ex-post and ex-ante) can be complementary and are achievable through the use of lotteries. Furthermore, lotteries have been broadly used in matching markets [17, 16, 27, 7]. We refer the reader to Section 2 for a more in-depth exposition to the use of

fair lotteries in KEPs and other applications.

Based on [33], we formally define a probability distribution over feasible exchanges. To this end, it is useful to think of a feasible exchange plan as a subgraph of the KEP graph. We call $\mathcal{F}_G$ the set of admissible subgraphs of G , *i.e.*, subgraphs that correspond to feasible exchange plans. We will use the notation $V(S)$ for $S \in \mathcal{F}_G$ to mean the vertex set of subgraph S .

Definition 1. A lottery is a probability distribution $\delta : \mathcal{F}_G \rightarrow \mathbb{R}_{\geq 0}$ over feasible exchanges for G . The set $\mathcal{F}_G(v) \subseteq \mathcal{F}_G$ denotes the set of feasible exchanges including pair $v \in P$ with $\delta_v = \sum_{S \in \mathcal{F}_G(v)} \delta(S)$ being the probability given to v by δ .

If we have such a probability distribution, we can draw a feasible exchange plan from it and implement this exchange plan in practice. Thus, pairs have a certain probability of being selected. Although the concept of utility for each transplantation $(i, j) \in A$ can mean various things (*e.g.*, expected survival time after transplantation), in this article, we will consider the probability of receiving a transplant under a particular distribution over feasible exchanges. This is motivated by the fact that most programs privilege the number of transplants, *e.g.*, see Figure 1 in [14] compiling the objectives of KEPs in Europe. However, our methodology is applicable to any utilitarian function. Going back to our example in Figure 1, we see that there are multiple ways to select a cycle of an exchange plan. We could consider the following two feasible exchange plans:

$$\begin{aligned} S_1 &= (V_1, A_1) & \text{where } V_1 &= V \setminus \{v_4\} & A_1 &= \{(v_1, v_2), (v_2, v_3), (v_3, v_1), (v_7, v_6), (v_6, v_5)\} \\ S_2 &= (V_2, A_2) & \text{where } V_2 &= V \setminus \{v_2, v_3\} & A_2 &= \{(v_4, v_1), (v_1, v_4), (v_7, v_6), (v_6, v_5)\}. \end{aligned}$$

We remark that S_2 does not contain a maximum number of vertices of P , but it includes vertex v_4 . It can make sense to consider such a sub-optimal feasible exchange plan since it may represent a chance of being transplanted for a patient of a pair that is otherwise excluded in the optimal solution. A combination of the two solutions given by $\delta(S_1) = p, \delta(S_2) = 1 - p$ with $p \in [0, 1]$ would give a probability $1 - p$ to vertex v_4 and probability p to vertices v_2 and v_3 of being selected in an exchange plan. Depending on the characteristics of the vertices that are selected, one solution could be considered fairer than the other. Furthermore, the process of improving v_4 's selection probability can be considered a fairness-enhancing approach.

5. Fairness schemes

In this section, we describe the various fairness schemes that we will compare for KEPs, *i.e.*, procedures to determine a lottery δ . We will focus on the most well-known and foundational approaches [10].

5.1. Mathematical description of fairness schemes

To discuss the concept of a fairness scheme [10] in the context of KEPs, we first introduce a useful definition.

Definition 2. Given an optimization problem of the form $\max\{\sum_{i=1}^m f_i(x, y) : (x, y) \in X \subseteq \mathbb{R}^{n_1+n_2}\}$, where $f : \mathbb{R}^{n_1+n_2} \rightarrow \mathbb{R}^m$ and $\{1, \dots, m\}$ is a set of agents, we define the **utility set**

$$U := \{u \in \mathbb{R}^m \mid \exists (x, y) \in X \text{ s.t. } f(x, y) = u\}.$$

In the context of KEPs, our set of agents are the patients and the optimization problem in Definition 2 that we will use to build the utility set is going to be (HPIEF) with

$$f_v(x, y) = \sum_{l \in V} \sum_{(i,v) \in A^l} \sum_{k \in \mathcal{K}(i,v,l)} w_{iv} x_{ivk}^l + \sum_{(i,v) \in A} \sum_{k \in K'(i,v)} w_{iv} y_{ivk}$$

for each $v \in P$ (here, $m = |P|$; we also slightly abused notation and index the function components with P). Again, variables x and y serve the same role as in (HPIEF). By default, we use $w_{iv} = 1$, *i.e.*, $f_v(x, y)$ is 1 if vertex v receives a kidney and 0 otherwise. We will also need the definition of a *fairness scheme*.

Definition 3. A *fairness scheme* is any function $\mathcal{L} : 2^{\mathbb{R}^m} \rightarrow \mathbb{R}^m$ that takes as input a utility set U for constraint set X and returns a utility vector. The notation $2^{\mathbb{R}^m}$ refers to the power set of $\mathbb{R}^m$, *i.e.*, the set of all subsets of $\mathbb{R}^m$.

In practice, a fairness scheme often takes the form of some process or algorithm that selects a solution among the set of feasible solutions. This latter set is given implicitly through the input U , which describes the set of attainable utilities given the constraints X (either ex-post utilities or the ex-ante expectation of utilities). In the typical setting where we maximize the number of transplants in a KEP exchange plan, we could see the function $\mathcal{L}$ as returning the utilities of an exchange plan maximizing said objective. Conceptually, this utilitarian objective is a fairness scheme. In spite of that, the goal of this work is to consider other fairness notions relevant to KEPs and afterward, derive a scheme balancing those notions and the utilitarian criterion.

When considering notions of fairness in a KEP, it is highly relevant to have some measure of the performance of the fairness scheme in terms of its cost to the maximum number of transplants, *i.e.*, the cost to utility. For this, Dickerson et al. [31], Bertsimas et al. [10] introduce the measure of *price of fairness* (POF).

Definition 4. The *price of fairness* of a fairness scheme $\mathcal{L}$ for a utility set $U \subseteq \mathbb{R}^m$ is given by

$$\frac{\sum_{i=1}^m u_i^* - \sum_{i=1}^m \mathcal{L}_i(U)}{\sum_{i=1}^m u_i^*},$$

where $u^* \in U$ is the maximum total utility that can be achieved as part of the set U , *i.e.*, $\sum_{i=1}^m u_i^* \geq \sum_{i=1}^m u_i \forall u \in U$, and $\mathcal{L}_i$ refers to the projection of $\mathcal{L}$ on component i . Note that $u^* = f(x^*, y^*)$ for $(x^*, y^*) \in X$, an optimal solution to the problem in Definition 2.

A POF close to 0 means that the fairness scheme results in a low sacrifice of the maximum utility. Conversely, if it is 1, then we are sacrificing all the utility in order to achieve fairness. In Figure 1, we can see that any fairness scheme resulting from the selection of S_1 , has POF equal to 0. The suboptimal exchange plan S_2 leads to a POF equal to 0.2 (S_2 gives a utility of 4 and the maximum utility from S_1 is 5).

5.2. Utilitarian principle

The *utilitarian principle* [64, 60, 10] is based on the idea that a fair allocation of resources maximizes social welfare, *i.e.*, the aggregation of the agents' utility. The most common notion of social welfare in KEPs involves the number of transplants.³ As mentioned in Section 5.1, it is worth pointing out that

³We could instead consider measures such as the expected number of years added to the patient's lifespan [44] or their associated quality adjusted life year [38, 39].

the utilitarian principle can be recast in the language of fairness schemes as it meets the requirements of the definition. However, so as to avoid confusion, from now on, we will distinguish the utilitarian principle from the other fairness schemes by not referring to it as such, unless explicitly stated.

5.3. Aristotle's equity principle

Aristotle's equity principle [2, 10] is based on the idea that when competing for access to resources, a fair allocation should consider pre-existing rights to these resources. For example, a patient that has waited a long time in the KEP pool should receive preferential treatment in order to obtain a kidney. This prioritization also has the effect of minimizing the deterioration of patients' health [20]. Other patient features can also be weighted in order to define the pre-existing right. In KEPs, the panel-reactive antibodies scores (PRAs) of patients are often considered as an indicator of priority. This score ranging between 0 and 100 measures the difficulty of matching a patient with a random donor from the population. A patient with a high PRA is called *hard-to-match*. The assumption is that the harder to match patients (pairs) will remain in the pool longer and thus may never be selected in time. Thus, the goal is to clear them from the pool as soon as there is a suitable match. In this scenario, the fairness scheme takes the form $\mathcal{L}^*(U) \in \arg \max_{\mathcal{L}} \sum_{v \in H} \mathcal{L}_v(U)$, where $H \subseteq P$ is the set of pairs with a PRA above a pre-determined threshold (*e.g.*, PRA above 80%) as used in Dickerson et al. [31], Gebel et al. [37]. This type of fairness notion is referred to as *group fairness* in the literature [33], and it has been complemented by works enlarging the utility set U (*e.g.*, Ashlagi et al. [5] emphasize the importance of considering long chains and cycles) and branch-and-price algorithms to more efficiently solve KEPs with expanded U [40]. The notion of priority can also be randomly determined, a mechanism referred to as *Random Serial Dictatorship* [28].

5.4. Nash standard of comparison

The *Nash standard of comparison* [49, 10] is based on the idea of n -player cooperative games. When $n = 2$, it measures the relative change in each player's utility when transferring a small amount of resources to the opponent. If the relative gain to one player is greater than the relative loss to the other, then this transfer of resources is justified. The same concept can be generalized to more than two players.

Definition 5. *Under the Nash standard of comparison (or proportional fairness), an allocation $\mathcal{L}(U)$ given by the fairness scheme $\mathcal{L}$ is fair if*

$$\sum_{i=1}^m \frac{u_i - \mathcal{L}_i(U)}{\mathcal{L}_i(U)} \leq 0 \quad \forall u \in U. \quad (1)$$

We measure the proportional change in utility when moving from $\mathcal{L}(U)$ to any other $u \in U$ and if the aggregate is always less than or equal to 0, then the allocation $\mathcal{L}(U)$ is considered fair. If we consider a starting allocation vector for the patients vertices, finding a proportionally fair allocation roughly translates to a transfer of utility from high-utility vertices to low-utility vertices. For instance, suppose that we start with a (deterministic) feasible exchange where some vertex v is selected (*i.e.*, $u_v = 1$), and some vertex v' is not selected (*i.e.*, $u_{v'} = 0$). Then, we determine a fairness scheme $\mathcal{L}(U)$, where only the utilities of v and v' are modified to $\mathcal{L}_v(U) = 0.95$ and $\mathcal{L}_{v'}(U) = 0.05$, *i.e.*, the probability of v and v' to be selected is 95% and 5%, respectively. This allocation $\mathcal{L}(U)$ would be justified. In fact, the 5% increase represents a significant improvement for the low-utility vertex, while a 95% chance of

transplantation is still remarkably high. It can be shown that the computation of a proportional fair scheme can be determined by maximizing the sum of the log-probabilities δ_v for the vertices (patients) included in some exchange plan. This is because the necessary and sufficient first-order optimality conditions for this solution correspond to the condition of the Nash standard of comparison [10]⁴.

5.5. Rawlsian justice

In Rawls [53], the author introduces the concept of the *veil of ignorance*. The key assumption is that every individual is not made aware of their own situation, biases, view of the world, etc. The only aspect that remains is knowledge about the world and their rational mind. Rawls argues that under this veil of ignorance, individuals can negotiate the terms of a social contract that seems inherently fair.

One way to approach this idea is to maximize the utility of the least well-off agent. When dealing with distributions over feasible exchange plans, this amounts to maximize the minimum probability of being in a selected exchange plan among the vertices in P of the KEP pool⁵. For this to make sense, we can assume that P does not contain vertices that cannot be part of any non-empty feasible exchange plan, *i.e.*, they need to be matched in some $\mathcal{F}_G$. Otherwise, the probability will always be 0, and we would obtain a skewed notion of fairness. Intuitively, the use of the minimum probability makes sense as it can directly be understood as the polar opposite to the utilitarian objective in a quantile optimization framework. Since we aim to understand the effect of fairness on the selection of exchange plans, Rawlsian justice provides the perfect candidate to jointly optimize with the utilitarian objective. It represents in a sense the tightest restriction to defining fairness in a quantile optimization framework.

As an example, we refer the reader to Figure 1. Using the Rawlsian approach, to maximize the minimum probability over the vertices, we could select the cycle (v_1, v_2, v_3, v_1) with probability $\frac{1}{2}$ and the cycle (v_1, v_4, v_1) with probability $\frac{1}{2}$. Here, we consider probabilities over feasible exchanges to give a chance to vertex v_4 .

5.6. Individual fairness

L_p -fairness for KEPs was first introduced by Farnadi et al. [33] as a measure of *individual fairness*. It provides a notion of distance to the mean probability of being selected as part of a distribution over exchange plans. The lower the value of this metric, the more the patient probabilities of receiving a transplant are equalized. If all patients receive the same probability of receiving a transplant, the value will be 0.

Definition 6. The L_p -fairness of a probability distribution δ is given by

$$\sum_{v \in P} (\delta_v - \bar{\delta})^p, \quad (2)$$

where $\bar{\delta} = \frac{1}{|P|} \sum_{v \in P} \delta_v$ is the mean probability of being selected as part of an exchange plan. A probability distribution δ is individually fair if it minimizes the Objective (2).

Following the definition of a fairness scheme, we can identify $\mathcal{L}_i$ with δ_v for the i^{th} vertex $v \in P$. The distribution δ that is individually fair implicitly defines a fairness scheme $\mathcal{L}(U)$ for the utility set U

⁴The first-order optimality condition for a concave function such as log is $f(y) \geq f(x) + \nabla f(x)(y - x)$.

⁵For a more general treatment using quantile maximization, see [55].

by the previous correspondence. In the original setting, the support of δ is restricted to exchange plans maximizing the number of transplants. Individual fairness is seen to generalize the previous concept of *Lorentz dominance* in pairwise KEPs [56, 45]. In this article, we do not restrict the support of our lottery δ as near-optimal exchange plans (in terms of utility) can be argued to be a good compromise with fairness. Moreover, we scale the use of individual fairness as the approach of Farnadi et al. [33] could only tackle small graph sizes.

6. Formulation and methodologies

Let us stress the goals of the fairness schemes for KEPs to be developed in this section: (i) to balance between utility and a fairness concept, (ii) to consider the chances of all patients to be in a selected exchange (ex-ante fairness), and (iii) to efficiently run the scheme in practice. To achieve (i), we will apply the Social Welfare Program (SWP) [36, 23] and the Nash Social Welfare Program (NSWP) [23, 52]. The solution to these programs is a point in the efficient set of the bi-objective problem consisting of the utilitarian objective and a fairness objective. In SWP, a linear combination of these objectives is optimized, while in NSWP, it is the area between a reference point and a point in the Pareto-front; a thorough overview of multi-objective optimization (MOO) concepts can be found in Appendix II. To attain (ii), we will define X as the set of distributions over feasible exchanges. Since all distributions are given over the feasible exchanges, it is possible for a distribution to have an exponentially large support. This justifies goal (iii). Indeed, it might be that a distribution optimal to our SWP and NSWP has a small support. Thus, we can instead seek to enumerate only the feasible exchange plans that will be part of the support of the optimal distribution. We will present a column generation method with this purpose. In this section, we lay out the conic programming formulations and column generation approach for the NSWP and note that the fairness schemes of Section 5 and the SWP are special cases of the NSWP (both are obtained from the NSWP with only one objective). Note that while we formalize fairness as an objective, we can also implement fairness constraints as described in Section 7.2.

For the sake of simplicity, in the following mathematical descriptions, we assume that the set V only includes vertices that can be matched in at least one feasible exchange plan, *i.e.*, $V = \cup_{S \in \mathcal{F}_G} V(S)$ (we abuse notation with the dual use of V both as a set and a function). In practice, this set is computed before running the column generation: for each vertex v , we manually set Constraint c_1 of (HPIEF) to 1 (*i.e.*, we turn it to an equality constraint) to force its inclusion in a solution. If one solution exists, v is considered in our set V , otherwise, we can safely remove it from the graph. Assuming that $S \in \mathcal{F}_G$ is a feasible exchange plan, the variable δ_S will denote the probability of selecting this exchange plan, *i.e.*, we use it as a short-hand for $\delta(S)$. We define $P' = V \cap P$ to be the set of pairs covered by at least one feasible exchange plan (remember the assumption on V mentioned above).

Aristotle's equity principle

One example of Aristotle's equity principle for KEPs involves the use of hard-to-match patients, *i.e.*, patients with a PRA above a certain threshold. Here, we use 80% as the threshold: every patient above this percentage is labelled as hard-to-match and in turn, its corresponding vertex (as done in Dickerson et al. [31]). We use the notation $V_H = P_H \cup N$, where $P_H = \{v \in P' \mid v \text{ has PRA} \geq 80\%\}$,

to denote the restriction of the vertex set V to pairs with high PRA (and in P') and NDDs.

$$\begin{aligned}
& \max r && (\text{P}_{\text{Aristotle}}) \\
\text{s.t. } & (\alpha_0) && \sum_{S \in \mathcal{F}_G} \delta_S = 1 \\
& (\alpha_1) && y_1 = \sum_{S \in \mathcal{F}_G} \sum_{v \in V(S) \setminus N} \delta_S - d_1 \\
& (\alpha_2) && y_2 = \sum_{S \in \mathcal{F}_G} \sum_{v \in V(S) \cap P_H} \delta_S - d_2 \\
& (\lambda_S) && \delta_S \geq 0 && \forall S \in \mathcal{F}_G \\
& (u \in \mathcal{Q}_r^3) && (y_1, y_2, r) \in \mathcal{Q}_r^3.
\end{aligned}$$

We denote dual variables in parentheses, while the computation of the reference point d is discussed in Appendix III. The object $\mathcal{Q}_r^3$ is the rotated second-order cone (see Appendix I for its definition). The constraint corresponding to dual variable α_0 ensures that we have a probability distribution over feasible exchanges. The variables y_1 and y_2 represent each multiplicative term in (NSWP): y_1 is the expected number of transplants minus d_1 (constraint with dual variable α_1) and y_2 is the expected number of hard-to-match patients in a selected exchange plan. Through $\mathcal{Q}_r^3$, we guarantee that the optimal objective value r coincides with $2 \cdot y_1 \cdot y_2$, which is equivalent to maximizing $y_1 \cdot y_2$ (constraint with dual variable u).

This problem formulation can have exponentially many variables δ_S , so feasible exchanges S will be generated as needed in a column generation fashion. To do so, we take the dual formulation of (P_{L_1}) :

$$\begin{aligned}
& \min -\alpha_0 + \alpha_1 d_1 + \alpha_2 d_2 && (\text{D}_{\text{Aristotle}}) \\
& \text{s.t. } && \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 1 \end{pmatrix} + u = 0 \\
& \alpha_0 - \sum_{v \in V(S) \setminus N} \alpha_1 - \sum_{v \in V(S) \cap P_H} \alpha_2 + \lambda_S = 0 && \forall S \in \mathcal{F}_G \\
& && u \in \mathcal{Q}_r^3 \\
& && \alpha \in \mathbb{R}^3, \lambda \geq 0.
\end{aligned}$$

Using $(\text{D}_{\text{Aristotle}})$, we immediately see from $\lambda_S \geq 0$ that we can rewrite the Constraints $(\dagger_S)$:

$$-\alpha_0 + \sum_{v \in V(S) \setminus N} \alpha_1 + \sum_{v \in V(S) \cap P_H} \alpha_2 \geq 0 \quad \forall S \in \mathcal{F}_G. \quad (\dagger_S)$$

In this way, given an optimal solution of $(\text{P}_{\text{Aristotle}})$ restricted to a subset of $\mathcal{F}_G$ (restricted master program), we can verify its optimality, or equivalently, dual feasibility, by finding the exchange plan minimizing the left-hand-side of Constraint $(\dagger_S)$. Thus, in Algorithm 1, we use Constraint $(\dagger_S)$ as the objective of the subproblem during the solution generation process. Indeed, the subproblem implements this objective within (HPIEF). Our methodology stops once the objective value of the subproblem is non-negative, *i.e.*, no Constraint $(\dagger_S)$ is violated. In essence, Algorithm 1 defines a

cutting plane procedure for the dual problem ($D_{\text{Aristotle}}$) and, consequently, a column generation for the primal ($P_{\text{Aristotle}}$). Since the problem of interest is ($P_{\text{Aristotle}}$), we refer to our methodology as a column generation.

In the remainder of this section, we do not provide details about the column generation approach for the next formulations given their similarity with the one just presented.

Algorithm 1 Decomposition algorithm to solve ($P_{\text{Aristotle}}$)

```

function MASTERPROBLEM( $G$ )
     $\mathcal{F}'_G \leftarrow \emptyset$  ▷ Initialize the set of solutions
     $\zeta \leftarrow -1$ 
    while  $\zeta < 0$  do
         $\mathcal{D} \leftarrow (D_{\text{Aristotle}})$  restricted to Constraints ( $\dagger_S$ ) for  $\mathcal{F}'_G$ 
         $(\alpha^*, \lambda^*, u^*) \leftarrow$  retrieve solution of  $\mathcal{D}$ 
         $S, \zeta \leftarrow \text{SUBPROBLEM}(\alpha^*, \lambda^*, u^*)$ 
         $\mathcal{F}'_G \leftarrow \mathcal{F}'_G \cup \{S\}$  ▷ Store new solution/column
    end while
    return  $\mathcal{F}'_G$  and the associated primal variables  $\delta^*$ 
end function

function SUBPROBLEM( $\alpha^*, \lambda^*, u^*$ ) ▷ Solve the subproblem
    Set  $w_{ij} = \alpha_1^* + \alpha_2^*$  for  $(i, j) \in A$  and  $j \in P_H$  in (HPIEF)
    Set  $w_{ij} = \alpha_1^*$  for  $(i, j) \in A$  and  $j \notin P_H$  in (HPIEF)
    Solve (HPIEF) and denote by  $S^*$  the optimal solution
     $V(S^*) \leftarrow \{v \in V \mid v \text{ is selected in the exchange plan } S^*\}$ 
    return  $S^*, \alpha_0^* - \sum_{v \in V(S^*) \setminus N} \alpha_1^* - \sum_{v \in V(S^*) \cap P_H} \alpha_2^*$ 
end function

```

Nash standard of comparison

When using the Nash standard of comparison, we seek to maximize the sum of the log-probabilities of being selected in an exchange plan, *i.e.*, $\sum_{v \in P'} \log \delta_v$. We then have the following (NSWP) formulated as an exponential conic program:

$$\begin{aligned}
 & \max r & (\text{P}_{\text{Nash}}) \\
 \text{s.t. } & (\alpha_0) & \sum_{S \in \mathcal{F}_G} \delta_S = 1 \\
 & (\alpha_1) & y_1 = \sum_{S \in \mathcal{F}_G} \sum_{v \in V(S) \setminus N} \delta_S - d_1 \\
 & (\alpha_2) & y_2 = T - d_2 \\
 & (\beta_v) & z_v = \sum_{S \in \mathcal{F}_G \mid v \in V(S)} \delta_S & \forall v \in P' \\
 & (\eta) & \sum_{v \in P'} t_v = T \\
 & (\lambda_S) & \delta_S \geq 0 & \forall S \in \mathcal{F}_G \\
 & (w_v \in (K_{\text{exp}})^*) & (z_v, 1, t_v) \in K_{\text{exp}} & \forall v \in P' \\
 & (u \in \mathcal{Q}_r^3) & (y_1, y_2, r) \in \mathcal{Q}_r^3,
 \end{aligned}$$

where K_{exp} is the exponential cone (defined in Appendix I). The variable z_v corresponds to the probability that v is selected in an exchange plan (constraint with dual variable β_v). The cone K_{exp}

ensures that the variable t_v is equal to $\log z_v$ in order to compute the sum of the logarithmic terms. The dual formulation (D_{Nash}) can be found in Appendix V.

Rawlsian justice

Using the Rawlsian approach, we aim to balance the utilitarian principle with the minimization of the probability of a vertex to be selected in an exchange plan. The corresponding (NSWP) formulation is

$$\begin{array}{ll}
& \max r \\
\text{s.t. } (\alpha_0) & \sum_{S \in \mathcal{F}_G} \delta_S = 1 \\
(\alpha_1) & y_1 = \sum_{S \in \mathcal{F}_G} \sum_{v \in V(S) \setminus N} \delta_S - d_1 \\
(\alpha_2) & y_2 = T - d_2 \\
(\beta_v) & z_v = \sum_{S \in \mathcal{F}_G | v \in V(S)} \delta_S & \forall v \in P' \\
(\gamma_v) & z_v \geq T & \forall v \in P' \\
(\lambda_S) & \delta_S \geq 0 & \forall S \in \mathcal{F}_G \\
(u \in \mathcal{Q}_r^3) & (y_1, y_2, r) \in \mathcal{Q}_r^3.
\end{array} \tag{P_{\text{Rawls}}}$$

Variable T will be made equal to the minimum of variables z_v (vertex probabilities of being selected in an exchange plan) through the constraints identified with dual variables γ_v . The dual formulation (D_{Rawls}) can be found in Appendix V.

Individual fairness

When applying the principle of individual fairness to KEPs, we make use of the L_1 norm as suggested by Farnadi et al. [33]. An individually fair scheme based on the L_1 -fairness minimizes this metric. Therefore, when in conjunction with the utilitarian principle, (NSWP) becomes the following

conic quadratic program

$$\begin{array}{ll}
& \max r \tag{P_{L_1}} \\
\text{s.t. } (\alpha_0) & \sum_{S \in \mathcal{F}_G} \delta_S = 1 \\
(\alpha_1) & y_1 = \sum_{S \in \mathcal{F}_G} \sum_{v \in V(S) \setminus N} \delta_S - d_1 \\
(\alpha_2) & y_2 = -T - d_2 \\
(\alpha_3) & |P'|z = \sum_{S \in \mathcal{F}_G} \sum_{v \in V(S) \setminus N} \delta_S \\
(\beta_v) & z_v = \sum_{S \in \mathcal{F}_G | v \in V(S) \setminus N} \delta_S - z \quad \forall v \in P' \\
(\lambda_S) & \delta_S \geq 0 \quad \forall S \in \mathcal{F}_G \\
(u \in \mathcal{Q}_r^3) & (y_1, y_2, r) \in \mathcal{Q}_r^3 \\
(w_v \in \mathcal{Q}^2) & (t_v, z_v) \in \mathcal{Q}^2 \quad \forall v \in P' \\
(\eta) & \sum_{v \in P'} t_v = T,
\end{array}$$

where the variable y_1 is the expected number of transplants minus d_1 (constraint with dual variable α_1). The variable y_2 is the negative of the L_1 metric minus d_2 since we are minimizing this loss (constraint with dual variable α_2). The object $\mathcal{Q}^2$ is the second-order cone (see Appendix I for its definition). The variable z is equal to the average pair probability of being selected in an exchange plan (constraint with dual variable α_3) and z_v is the difference between δ_v and z (constraint with dual variable β_v ; recall Definition 1). Through $\mathcal{Q}^2$, we ensure that t_v is equal to the absolute value of z_v (constraint with dual variable w_v).

We can now easily adapt the methodology described for NWSP-based schemes to SWP-based schemes. If we are only interested in the fairness schemes of Section 5, we can simply replace the constraints with the dual variable α_1 to $y_1 = 1$ in all the formulations provided in this section. If we are interested in the SWP, we can replace r by $\lambda_1 \cdot y_1 + \lambda_2 \cdot y_2$, where λ_1 and λ_2 are given scalars, and remove the constraints associated with the dual variable u . In Appendix IV, we provide a solved examples for individual fairness schemes other than Rawlsian justice (since this case is covered in this section).

7. Dual analysis

Next, we provide an interpretation of the solution to the dual problems presented in Section 6 in term of vertices' weights. This enables us to draw a distinction between deterministic policies based on the optimization of a (weighted) linear objective and distributions over exchange plans. We focus on the dual of (P_{Rawls}) , but an analogous analysis is possible for the other formulations.

7.1. Interpretation of the dual coefficients

By complementary slackness and strong duality applied (P_{Rawls}) and (D_{Rawls}) , we have that:

Lemma 1. Let $(\delta^*, y^*, z^*, r^*)$ be an optimal solution to problem (P_{Rawls}) and $(\alpha^*, \beta^*, \gamma^*, \lambda^*, u^*)$ be an optimal solution to problem (D_{Rawls}) . For each $S \in \mathcal{F}_G$ such that $S \in \text{supp}(\delta^*)^6$, $\lambda_S^* = 0$ and

$$\alpha_0^* - \sum_{v \in V(S) \setminus N} (\alpha_1^* + \beta_v^*) = 0, \quad (3)$$

where the summation on the left-hand side of Equation 3 can be written as

$$- \sum_{v \in V(S) \setminus N} (\alpha_1^* + \beta_v^*) = -\alpha_1^* |V(S) \setminus N| - \sum_{v \in V(S) \setminus N} \beta_v^*. \quad (4)$$

For all $S \in \mathcal{F}_G \setminus \text{supp}(\delta^*)$, the left-hand side of Equation 3 is strictly smaller than 0.

By observing the right-hand side of the Equation 4 we see that the first term is the weight of the solution that is given to the overall utility, while the other term is the sum of individual weights. Therefore, the constraints given by Equation 3 can be understood as a weighted sum of global utility scores and individual scores. This provides a natural interpretation for the selection of particular solutions as part of the support of the optimal distribution. We underscore that the optimal weights that are recovered from the dual depend on the initial graph. Of note, we can see that β_v^* in (D_{Rawls}) is equal to 0 by complementary slackness, unless vertex v is attributed the minimum transplant probability among vertices from P' ; this allows us to identify the patient-donor pairs that are hard-to-match according to the graph structure instead of relying on pre-established criteria such as PRA.

7.2. Weights, support and constrained distribution

While the weights that we compute using the duals define a hyperplane containing all solutions from the support of the optimal distribution, they do not provide any probability of selecting $S \in \text{supp}(\delta^*)$. Moreover, any approach that attempts to obtain weights from data for each cycle or arc (depending on the KEP formulation) will only manage to define a similar hyperplane of optimal KEP exchange plans. This hyperplane does not provide any information as to how to select an exchange plan from it. Nevertheless, we wish to point out that the patient probabilities recovered using our method can be used to define weights that replace or even complement expert-defined weights based on clinical information. The probabilities encode information related to the structure of the KEP graph and provide valuable information about the compatibility of certain patients with the rest of the pool, which can be missing in typical weight-based approaches.

We discussed above that lotteries allow us to select among optimal solutions given a deterministic weighted objective, and they can also be regarded as defining weights. Finally, we remark that besides their ex-ante guarantees, we can also obtain ex-post guarantees. First, any exchange plan S drawn from δ^* is optimal to a weighted objective (Lemma 1). Second, we can add constraints to our subproblem in Algorithm 1, and thus restrict the exchange plans considered in the support of the lottery, without any further modification to the algorithm; note that the subproblem solved in each iteration only has modified objective coefficients but its constraints are unaltered.

⁶The notation $\text{supp}(\delta)$ refers to the support of probability distribution δ .

8. Evaluation

Next, we analyze multiple experiments designed to evaluate the benefits of introducing the fairness schemes in Section 5 and the fairness schemes based on the SWP and the NSWP. Prior to this, we describe our computational environment, instances and tested schemes. Based on the experimental results from this section, we make practical guidelines for decision-makers that are summarized in Appendix IX.

8.1. Experimental setup

Each experiment was run on one cluster node powered by two Intel E5-2683 v4 Broadwell CPUs running at 2.1Ghz. A time limit of 1 hour per instance was set, and 16GB of memory was used. All the code was implemented in Julia 1.7.1 and optimization problems were solved with Mosek 9.2.⁷, with a tolerance parameter of 10^{-8} when solving the NSWP.

Based on the Saidman generator [58], Dickerson et al. [30] created graph instances (including PRA) that are available in Preflib [48]. From this repository, we used 10 instances for each combination of $|P| \in \{16, 32, 64, 128, 256\}$ and percentage of NDDs in $\{0\%, 5\%, 10\%, 15\%\}$, except for $|P| = 16$ and 5% NDDs, where no instance was considered. These sizes range over some KEP pool sizes used in Europe [12] and Canada [19]. The cap limit for cycles and chains length was set to $K = 3$ and $K' = 3$. These limits are consistent with some existing KEPs [12], and increasing them would mainly augment the set of feasible exchange plans available for our fairness schemes.

We did not consider larger instances as we either did not have enough computing power or memory to build the (HPIEF) formulation. Note that the efficiency of our decomposition method goes hand in hand with the state-of-the-art (integer programming) approaches for KEPs. These correspond to the subproblems in our column generation procedure (recall Algorithm 1), whose improvement is beyond the scope of this article.

8.2. Results

In the experimental results, we abbreviate to Utilitarian, Nash, Aristotle, Rawls and IF, the fairness schemes considering the utilitarian principle (number of transplants), Nash standard of comparison, Aristotle’s equity principle, Rawlsian justice and individual fairness, respectively. We use the term “Single” when a scheme is based on the optimization of a single fairness concept, “NSWP” when the NSWP scheme is considered and “SWP” when the objective is a linear combination of a fairness concept and the utilitarian objective. The weights for the linear combination were chosen as $(\lambda_1, \lambda_2) = \left(\frac{1}{i_1 - d_1}, \frac{1}{i_2 - d_2}\right)$ in order to have similar scales for both objectives, with i and d being the ideal and nadir point, respectively. There is an exception to this rule for Nash, where we use a different d . See Appendix III for details on the computation of d .

The set of research questions that we investigate can be divided into two groups. Research questions 1, 2, and 3 analyze the effect of combining a utilitarian objective with a fairness objective. We analyze how effective the methods are in terms of the quality of the solutions with respect to each fairness objective. Research questions 4 and 5 empirically measure the efficiency of the column generation approach (Algorithm 1) in practice. We provide complementary research questions in Appendix III and VI.

⁷www.mosek.com

RQ1: What is the effect of different fairness schemes on utility? First, we compute the average POF of each fairness scheme over all instances. These values can be found in Figure 2. Only instances for which all schemes were solved within the time limit were considered (details provided in Appendix VIII).

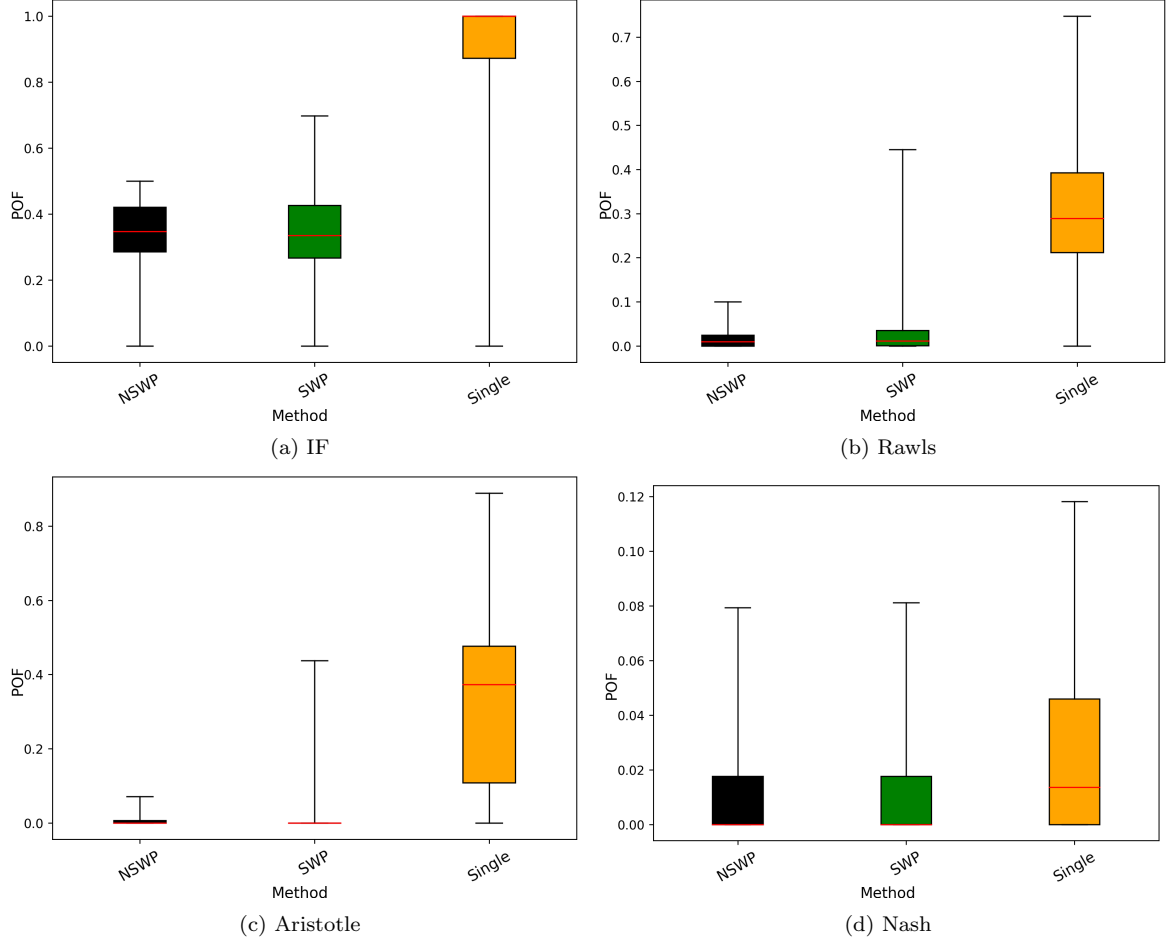


Figure 2: Comparison of SWP, NSWP and Single in terms of POF

As expected, the Single schemes have the highest POFs in comparison with SWP and NSWP since the number of transplants is not considered. This supports that schemes based solely on fairness may not be adequate, since they may significantly decrease the utilitarian value. Overall, the NSWP has on average very similar POFs to that of the SWP schemes. The POFs are low indicating that fairness does not come at the cost of utility. IF is the only exception to the similarity between SWP and NSWP. However, we note that this is a byproduct of the fact that the empty (or a non-maximal) exchange plan may be selected in NSWP: for example, the empty exchange plan would obtain an optimal score of 0 for the fairness objective and would possibly be included in the support (see Appendix VII for a solution based on bilevel optimization that eliminates non-maximal exchange plans). A more practical approach would consist of adding a constraint to remove any exchange plan with a total number of patients below a threshold. For details on how to implement this idea, we refer the reader to Section 7.2.

RQ2: What is the effect of the utilitarian approach on the fairness objectives? Given the good POF of the SWP and the NSWP, in this question, we focus on those schemes. We looked at the value of the fairness objective for the NSWP, SWP and the utilitarian approach. The goal is to

observe how much of an improvement is obtained by the NSWP or SWP in the fairness value when compared to the standard utilitarian method. The values can be found in Figure 3. Again, we only consider instances for which we were able to apply the three approaches within the time limit.

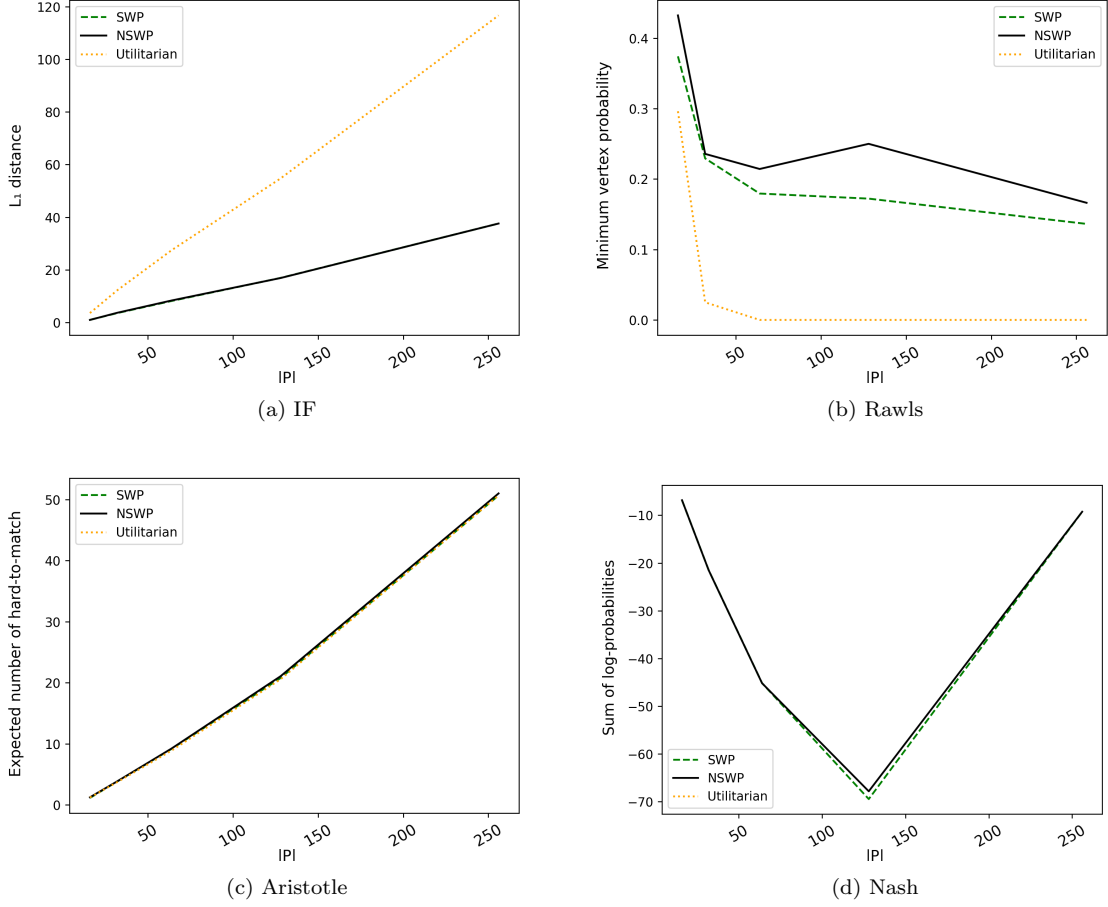


Figure 3: Comparison of SWP, NSWP and the utilitarian approach in terms of the value for each fairness objective

A marked difference can be observed for IF and Rawls, where the NSWP and SWP show a significant improvement to the fairness objective. However, we do not see any difference for Aristotle and Nash’s fairness principles. The Aristotelian case is made obvious by our analysis of the support size in RQ5. Essentially, for many instances there is an exchange plan achieving the ideal values. For Nash’s fairness objective, we averaged the values that did not return negative infinity, a recurrent property for the utilitarian scheme as some vertices are not included in the support of the NSWP and SWP. This is the reason why Utilitarian is not displayed on Figure 3 for Nash.

RQ3: What is the minimum number of pairs included in exchange plans from the support? For this question, we investigate how acceptable the selection of an exchange plan would be to patients affected by our methodology. Given a lottery, *i.e.*, a probability distribution, we look at the exchange plans with strictly positive probability. Among those, we compute the minimum number of pairs that are part of a single exchange plan. Figure 4 reports the values obtained as a fraction of the the maximum number of pairs that can be included as part of the utilitarian approach.

We can observe that the SWP and the NSWP perform similarly in terms of the minimal fraction of patients that are included among the solutions in the support (*i.e.*, with strictly positive probability). Both provide a noticeable improvement over the single objective case for IF, Rawls and Aristotle. The

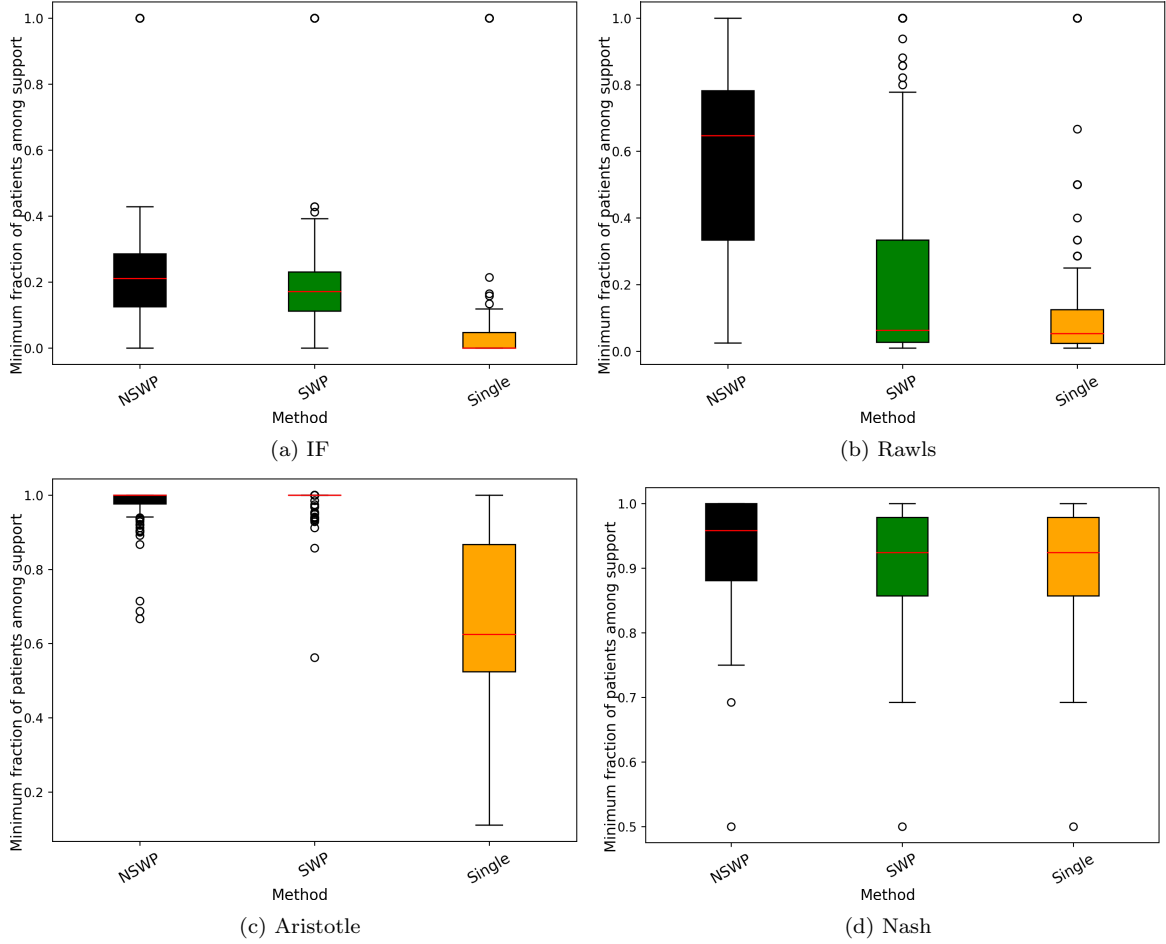


Figure 4: Comparison of SWP, NSWP and Single in terms of included vertices

three approaches exhibit the same minimal size of exchange plan in their support. The main takeaway is that both SWP and NSWP always perform at least as well as the single objective case on average, with a noticeable improvement for every scheme except Nash.

RQ4: How scalable are our schemes? We compute the running time to solve Single, Linear and NSWP using our column generation approach. In Figure 5, we report performance profiles and in Figure 6, we provide the running times averaged over the various sizes of P for each scheme.

In all the fairness schemes, the curves for SWP and NSWP (see Figure 5) are very close and follow similar trajectories. Figure 6 underscores the same point: the time taken by SWP and NSWP is distributed very similarly. Both approaches are not empirically costly to run when compared to the single objective setting and given that this is a planning problem.

A key aspect to consider when evaluating the scalability of our proposed methods is the time spent solving the subproblem. Since the SWP and the NSWP are column generation methods, the time spent in the subproblem should form the main computational bottleneck for this method to be viable. In Figure 7, we report the fraction of the total time spent solving the subproblem for each scheme and $|P| = 128$. From Figure 7, we can observe that often more than 90% of the computational time is spent in the subproblem for IF and Nash. For Rawls and Aristotle, these values are also higher than 50%, and often reach above 80% for Rawls. We also included other values $|P|$ in the appendix. These results reflect that any improvements to MIP solving abilities will in turn benefit our approach. There is already a rich literature on the subject (*e.g.*, [54, 3]) and we note the possibility of already

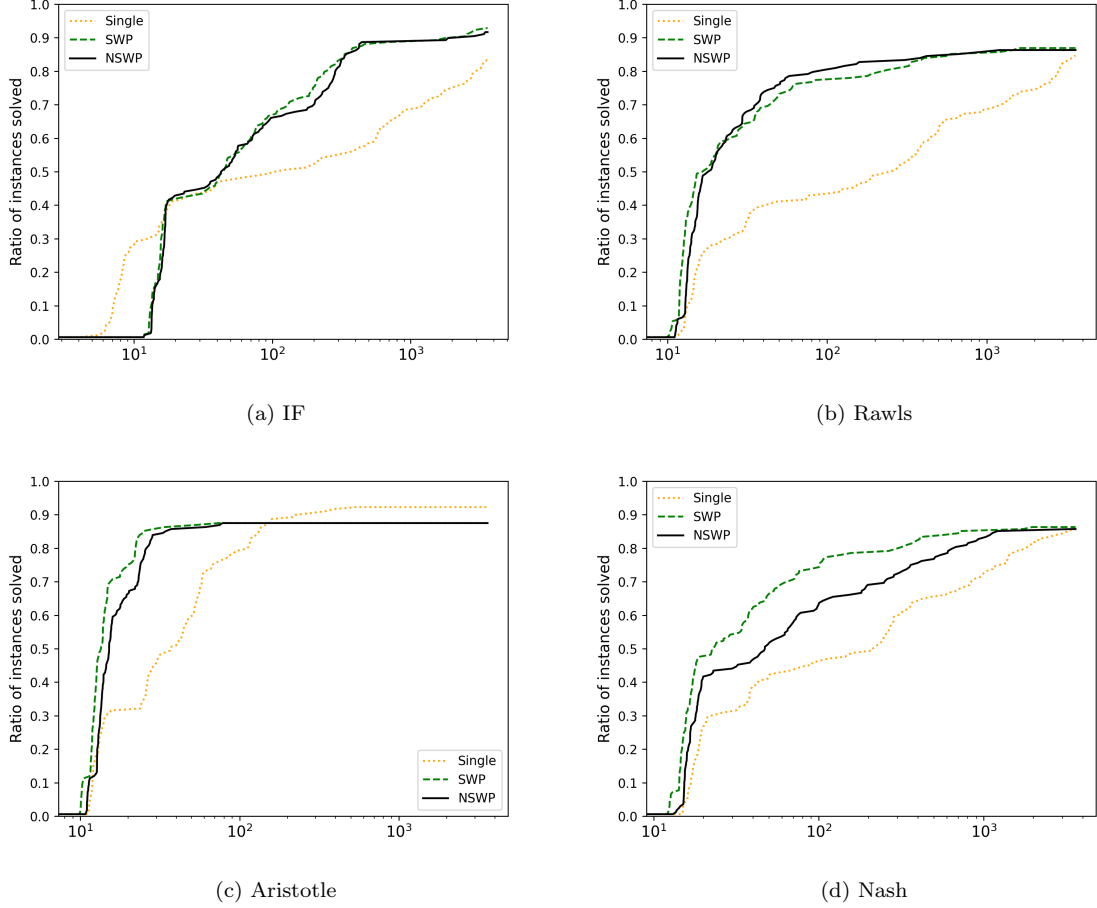


Figure 5: Ratio of instances solved for each model over time

existing methods leveraging the structure of KEPs to greatly improve the scalability. For example, in the case of $K = 2$, polynomial-time methods for tractable kidney exchange problems exist, as well as in the case of unbounded exchanges [1, 8]. In these cases, the methodology described in Section 6 involves a subproblem that is solvable in polynomial time. The master program is a conic program, which can be solved using either the ellipsoid method or an interior-point method. Consequently, the overall method can run in polynomial time, as the equivalence between optimization and separation (formalized in [41]) ensures that the procedure remains efficient.

RQ5: How many feasible exchange plans are part of the support of the optimal distribution? We are interested in determining the number of feasible exchange plans in the support of the lotteries for our fairness schemes. This will complement our understanding of their running times.

We do note that the enumeration of exchange plans (columns) is usually stopped when there is a marginal increase to the objective (using our tolerance parameter). Therefore, we only have an approximation to the optimal distribution, but this is to be expected in any case because we are dealing with numerical methods. In Figure 8, we provide the average number of exchange plans in the support over all graph sizes for the three optimization approaches. We can see that the support size is similar for all schemes (with a caveat for IF). The support size for Single is lower than NSW and SWP for IF and Rawls. This intuitively makes sense as the multi-objective setting needs to include solutions that are in the support of the optimal distribution with respect to each objective. Schemes using Aristotle’s objective have support size typically equal to 1, because it is often possible to have

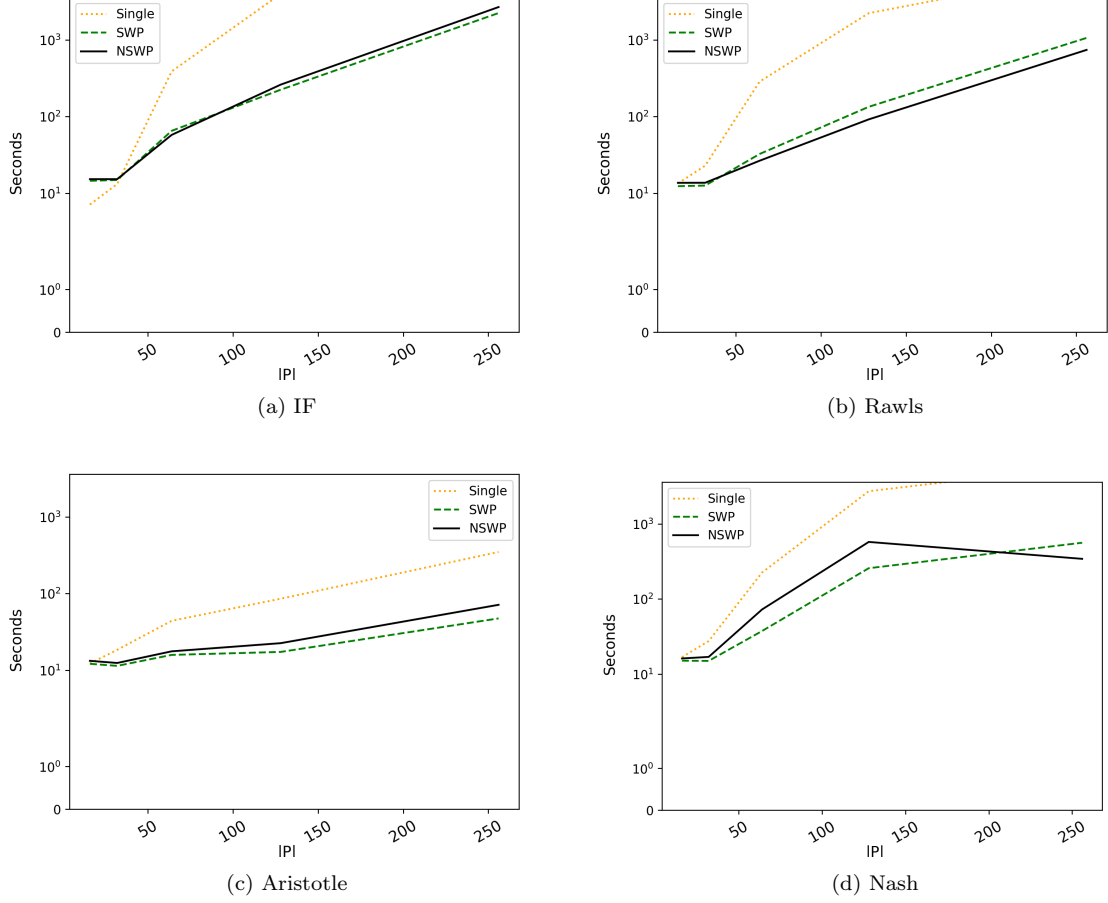


Figure 6: Time to solve the models for each scheme

a single solution that includes the maximum number of transplants and the maximum number of hard-to-match patients.

Final remarks.

After setting out to answer the prior research questions, we can conclude that both the SWP and the NSWP perform similarly with respect to POF, computational time, and lottery support size. In addition, for most solvable instances by Single, both bi-objective approaches were applicable in the allocated time (see Appendix VIII). In both IF and Rawls, the NSWP has better ex-ante guarantees in terms of the utilitarian objective, while for Aristotle, the SWP has better ex-post guarantees (see Figure 3 and 4). Please see Appendix VI for further experiments evaluating the performance of a chosen (*i.e.*, optimized) fairness objective in terms of other fairness scores (*e.g.*, optimizing for Rawlsian justice and evaluating its performance with respect to Nash’s standard of comparison).

9. Conclusions and future work

In this work, we have shown that fairness in KEPs inherently involves a tradeoff between maximizing the number of transplants (a common choice of utilitarian objective; see Ashlagi et al. [4]) and enhancing the chance of being selected in an exchange plan for patients that would otherwise be overlooked. We attempted to bridge these two dichotomies by applying the SWP and NSWP. We implemented this idea using various fairness schemes and provided a column generation for their efficient use. Our empirical results showed that the schemes based on the SWP and the NSWP fairness

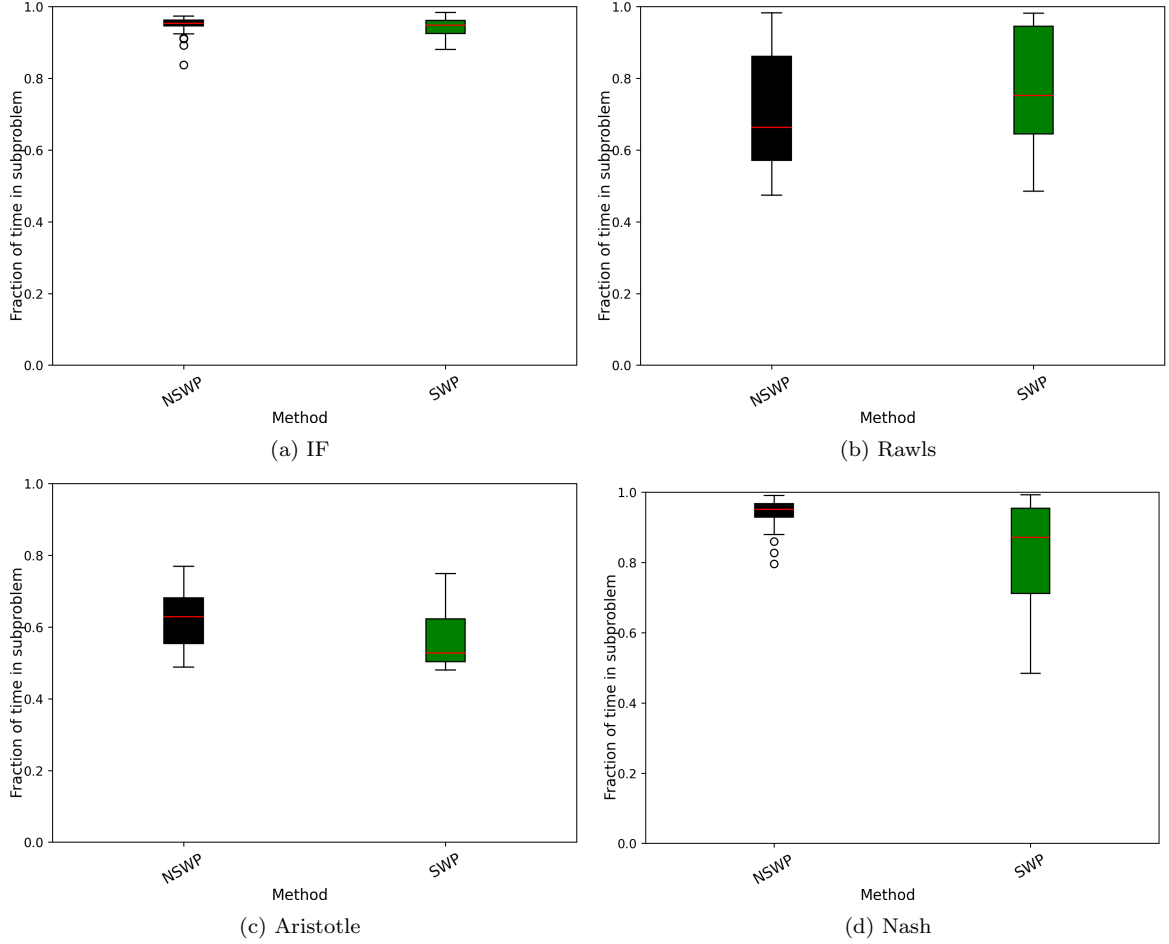


Figure 7: Time to solve the models for each scheme

schemes have a low POF and the worst-case utilitarian outcome is higher than schemes based only on fairness concepts. This means that their practical application is likely to keep the utilitarian objective close to its maximum value while increasing the patients’ chances of being transplanted.

Our work contributes with a catalog and a systematization of fairness concepts for KEPs. It also provides a novel combination of fairness schemes with lottery policies for KEPs, with the potential to impact the way exchange plans are chosen in practice. In practice, we envisage that lottery policies, once approved by ethics committees, would be subject to a randomized draw. This process could be done, for example, through a live lottery as in Flanigan et al. [34] (here, lotteries are used for the selection of citizens’ assemblies). Most of the fairness metrics used in this paper do not consider clinical features of the patients, except for the Aristotle’s equity principle. The latter uses the patients’ PRA which is a value broadly used in practice and thus “accessible” to practitioners. The remaining fairness concepts rely on the mathematical topology of the KEP graphs. In future work, it would be of interest to go beyond this and consider individual features of the patients such as time in dialysis and health status, to name a few. We also envisage the need to account for other important factors in solving KEPs, such as including compatible patient-donor pairs and ABO-incompatible transplants [34]. In the latter case, ABO-incompatible transplants are more often unsuccessful and expensive than compatible ones. Thus, maximizing the number of transplants is not necessarily the most sensible choice for a utilitarian function. In our experiments, we only considered ABO-compatible pairs, but the inclusion of ABO-incompatible pairs would offer a more realistic setting for testing the

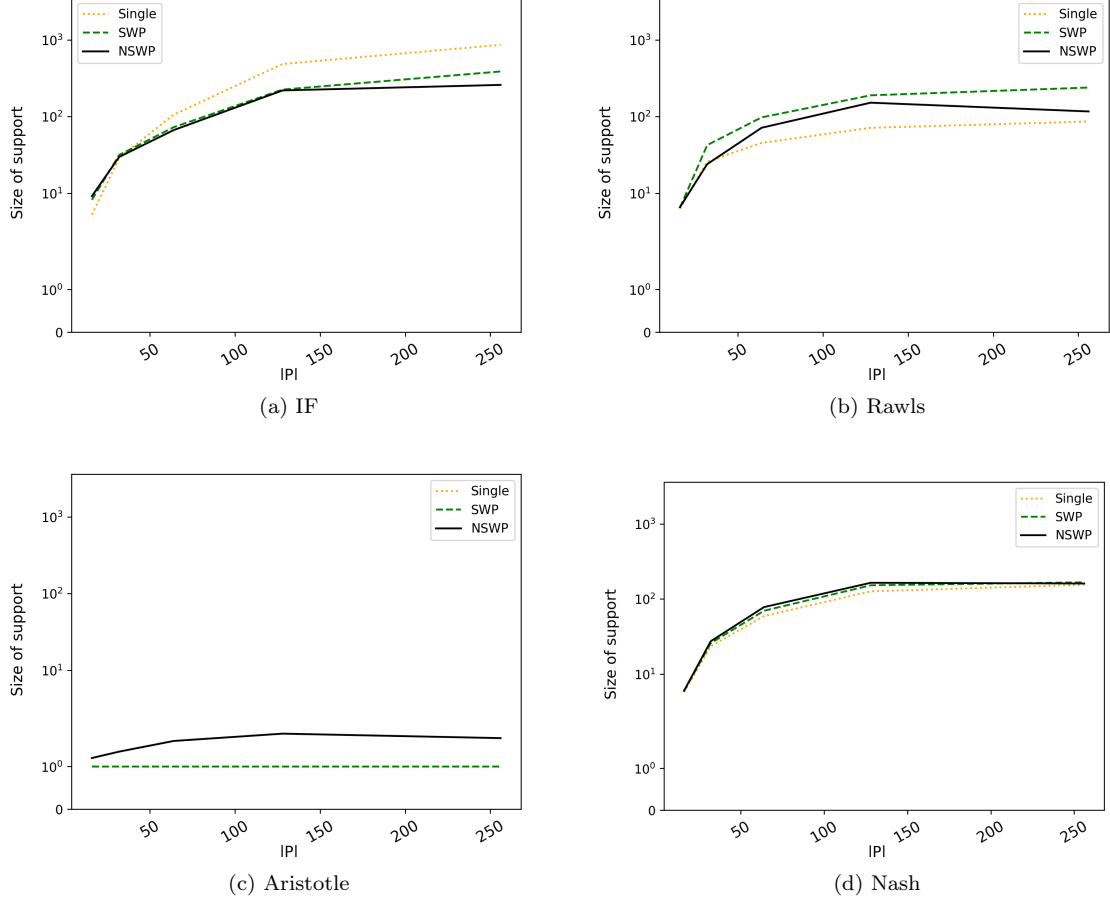


Figure 8: Size of the support for each scheme

viability of our method [12]. We did not consider the quality of proposed matchings in our experiments as we did not have access to such data, but in a real-life setting, this valuable information needs to be accounted for as it can severely impact the decision of patients to accept or reject a proposed donor [39].

The aforementioned variations to the setting explored in our article can all be accommodated using our framework subject to slight modifications to the objectives and the overall structure of the conic programs. Expanding on the decision-making aspect, we can envisage patients acting strategically and influencing the set of feasible matchings through their donor characteristics (*e.g.*, as in [56, 13]). This could potentially influence various fairness scheme approaches discussed in this work. For example, under a Rawlsian fairness scheme, a patient could exploit the maximization of the minimum selection probability by introducing a highly incompatible donor to ensure their inclusion in the matching. Furthermore, in the UK, for example, waiting time estimation is available to patients based on their donor characteristics [50]. It would therefore be possible to examine the weight that such information plays in a patient’s willingness to participate in a KEP governed by our fairness schemes. A thorough analysis of game theoretical notions for our fairness schemes is therefore valuable to every real-life implementation.

Moreover, future research must account for the fact that KEPs are not static and run over time. This increases the flexibility of applying fairness concepts that are currently myopic. The generalization of fairness schemes to dynamic KEPs will likely pose computational difficulties since it would require

the consideration of future feasible exchange plans which make the problem highly combinatorial. For examples involving simulations, see Santos et al. [59]. To improve on the quality of simulations for non-static KEPs, follow-up research should move to more realistic instances such as the ones present in Delorme et al. [26]. Furthermore, comparison with other methods that are concerned with transplant probability are warranted, such as the method in Keizer et al. [42] that counts for the number of transplants containing each individual patient and attributes a weight to each patient. As mentioned above, this is achieved through a modification of the objective and falls under the general notion of fairness schemes introduced in this work. Lastly, the improvement of individual fairness is an important line of study. For instance, the restriction of this fairness scheme to maximal exchange plans would likely address the issue surveyed in our experiments (see Appendix VII).

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Appendix

I. Conic programming definitions

Definition 7. The second-order cone $\mathcal{Q}^n$ is given by $\mathcal{Q}^n = \{x \in \mathbb{R}^n \mid x_1 \geq \sqrt{x_2^2 + \dots + x_n^2}\}$.

Definition 8. The rotated second-order cone $\mathcal{Q}_r^n$ is given by $\mathcal{Q}_r^n = \{x \in \mathbb{R}^n \mid 2x_1x_2 \geq x_3^2 + \dots + x_n^2, x_1, x_2 \geq 0\}$.

Definition 9. The exponential cone K_{exp} is defined by

$$K_{exp} = \left\{x \in \mathbb{R}^3 \mid x_1 \geq x_2 e^{\frac{x_3}{x_2}}, x_2 > 0\right\} \cup \left\{x \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 = 0, x_3 \leq 0\right\},$$

and its dual $(K_{exp})^*$ is given by

$$(K_{exp})^* = cl \left\{y \in \mathbb{R}^3 \mid y_1 \geq -y_3 e^{\frac{y_2}{y_3}-1}, y_1 > 0, y_3 < 0\right\}.$$

II. Background to MOO

Multi-objective optimization (MOO) concerns problems of the form

$$\max_{x \in X} (f_1(x), \dots, f_k(x)), \quad (\text{MOO})$$

where $X \subseteq \mathbb{R}^n$ and f_i for $i = 1, \dots, k$ are real-valued functions. Typically, we are interested in the computation of *Pareto-optimal* solutions.

Definition 10. A solution $x^* \in X$ is *Pareto-optimal* to (MOO) if there is no $y \in X$ such that (i) $f_i(y) \geq f_i(x^*)$ for all $i = 1, \dots, k$ and (ii) $f_j(y) > f_j(x^*)$ for some $j \in \{1, \dots, k\}$. The set of Pareto-optimal solutions is called the *efficient set* and the associated optimal values form the *Pareto front*.

In some instances of mathematical programming, it is possible to enumerate the full Pareto front [63], but distributions over exchange plans in KEPs can lead to exponentially many such solutions. For this reason, we explore MOO methods that can provide a simple rule to select from the Pareto front; they will serve as the main focus of this Section. In Section II.I, we will review the most natural approach to obtain solutions of the efficient set, called the Social Welfare Program, where we optimize a linear combination of objectives. In Section II.II, we move on to the Nash Social Welfare Program, which also provides Pareto-optimal solutions, and discuss why it might be interesting compared to the previous approach.

II.I. The Social Welfare Program

The Social Welfare Program (SWP) [36, 23] provides a framework in the context of multi-objective optimization to analyze the trade-off between the objectives, which in our case are utility and fairness. Concretely, a Social Welfare Program for (MOO) is any optimization problem of the form

$$\max_{x \in X} \sum_{i=1}^k \lambda_i f_i(x)$$

where $\lambda_i > 0$ for $i = 1, \dots, k$ are fixed weights.

From Definition 10, it is easy to see that a solution to an SWP is Pareto optimal for the respective multi-objective problem. Thus, by solving SWPs with different weights, various points of the Pareto frontier may be computed. However, there are inherent weaknesses that come with the SWP approach [23]. The main weakness with which we shall be concerned in the context of this article deals with the non-existence of a criterion (or consensus) for selecting a Pareto-optimal solution. In particular, it is not obvious how to set the weights λ to obtain a Pareto-optimal solution balancing our objectives. This weakness is a major one since we do not know how to compare two Pareto-optimal solutions. We refer the reader to Charkhgard et al. [23] for further details concerning weaknesses of the SWP and how they are addressed by the NSWP.

II.II. The Nash Social Welfare Program

The Nash Social Welfare Program (NSWP) [23, 52] is an approach addressing the selection of a Pareto frontier point. It interprets the objective functions in Problem MOO as the payoffs of k agents and it considers a reference $d_i \in \mathbb{R}, i = 1, \dots, k$ indicating to agent i the minimum payoff they can guarantee. Hence, $f_i(x) - d_i$ becomes the surplus of agent i for solution x , with the NSWP modelling the search for an $x \in X$ such that the multiplication of all agents' surplus is maximized. Mathematically, it has the form:

$$\begin{aligned} & \max \prod_{i=1}^k (f_i(x) - d_i)^{\lambda_i} \\ & \text{s.t. } x \in X \\ & f_i(x) \geq d_i \quad i = 1, \dots, k. \end{aligned} \tag{NSWP}$$

The hyper-parameters $\lambda_i > 0$ are chosen beforehand and represent the weights attributed to each objective as in the SWP. In dimension $k = 2$ with $\lambda = (1, 1)$, the NSWP attempts to maximize the total rectangular area defined by the reference point d and the solution picked from the Pareto front (see Figure II.1).

The NSWP possesses two key properties: it is both global-power-scale-free and local-benefit-scale-free. In other words, multiplying the terms $f_i(x) - d_i$ with the same coefficient (resp., multiplying the weights λ_i by the same coefficient) will result in an equivalent problem (up to the multiplication constant), *i.e.*, the optimal solutions will be the same.

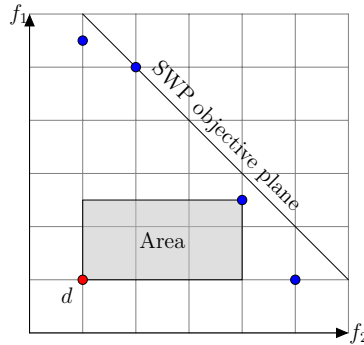


Figure II.1: Pareto front (blue dots) and the NSWP optimal objective (given by the gray area). The SWP optimal objective plane is depicted with $\lambda = (1, 1)$. Inspired by Figure 2 in [23]

III. Reference point

Next, we detail the computation of the reference point (d_1, d_2) , essential for the formulation of the NSWP schemes. We begin by giving a definition of the ideal and nadir points in MOO. We denote by X^* the efficient set.

Definition 11. Given a vector-valued function $f : X \rightarrow \mathbb{R}^k$, we define the ideal i to be $i^\top = \left(\sup_{x \in X^*} f_1(x), \dots, \sup_{x \in X^*} f_k(x) \right)$.

Definition 12. Given a vector-valued function $f : X \rightarrow \mathbb{R}^k$, we define the nadir n to be $n^\top = \left(\inf_{x \in X^*} f_1(x), \dots, \inf_{x \in X^*} f_k(x) \right)$.

In order to obtain the reference point for our NSWP models with respect to IF, Rawls and Aristotle, we partially build the models without including the terms d_1, d_2 and we use only y_2 as an objective. We can use Algorithm 1 to generate exchange plans that will lead to the best f_2 score possible. We then set this value to be i_2 . We add the temporary constraint $y_2 = i_2$, replace the objective by y_1 and solve the mathematical program, obtaining d_1 . Afterwards, we replace the temporary constraint by $y_1 = i_1$, where i_1 is the maximum number of expected transplants that can be performed. Solving the model with objective y_2 , we obtain d_2 . Thus, the nadir point is given by (d_1, d_2) (in the schemes that use it, *i.e.*, IF, Rawls, and Aristotle), while the ideal is (i_1, i_2) .

Nash's principle of fairness is slightly different. Since it sometimes occurs that some vertices are never included by a solution that maximizes the number of transplants, the value d_2 would be negative infinity. This is a problem and we must therefore rely on an ad hoc notion. This notion gives the worst-case scenario that the best value of f_2 can take over all graphs. In other words, if we have a complete bipartite graph $G = (V, A)$, where $V = P \cup N$ with $N = \{1\}$ and $P = \{2, \dots, n+1\}$, we can see that the only possible exchanges involve selecting vertex 1 and one from P . In this case, the optimal solution selects the vertices from P with probability $\frac{1}{n} = \frac{1}{|P|}$. Using this fact, we can set d_2 to be equal to $-|P| \log(|P|)$ (note that we use P' in our experiments).

We can add the values of d_1 and d_2 to the right constraints and finalize the construction of our mathematical programming models. The exchange plans that were generated during this whole process can be used as a warm start to our set of solutions for Algorithm 1 when maximizing the NSWP models with objective r .

In all experiments involving IF, Rawls or Aristotle, the nadir point is used. In order to empirically justify the use of this reference point instead of another arbitrary point, we evaluate the POF of SWP and NSWP with Rawls. We do this both for the nadir point and the point $(0, 0)$. We report the distributions of POF values in Figure III.1.

From Figure III.1, we can observe that the nadir point leads to a better POF than the point $(0, 0)$. Intuitively, the nadir point takes into account meaningful information about the "worst-case scenario" for each objective. The point $(0, 0)$ neglects the potential magnitudes of the objectives, and is therefore potentially sensitive to the differing scales. These preliminary results lead us to use the nadir point. However, we underscore that any reference point can be used within our framework.

A valuable indicator to evaluate the scalability of our approach is the time spent computing the reference point (*i.e.*, the nadir). Some large instances are hard to solve exactly and even the utilitarian case would prove difficult to handle. Since we care about lottery policies over exchange plans, solving

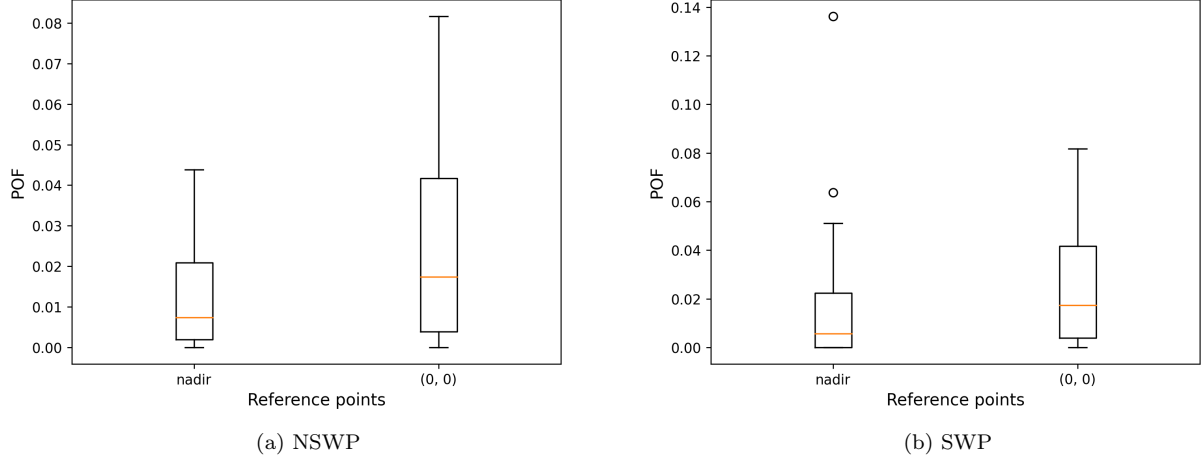


Figure III.1: Comparison of the distributions of POF values for the nadir point and $(0,0)$ for SWP and NSWP with Rawls

these challenging instances will be harder. In order to better understand the fraction of the total computation time spent in obtaining the reference point, we refer the reader to Figure III.2. We see that as the instances become larger and harder to solve, a significant portion of the computation is spent in obtaining the reference point of Rawls and Aristotle, especially, the second component (*i.e.*, the fairness objective) of the reference point. In these cases, the column generation procedure is very efficient (when we observe the total running time of Figure 5): solving the subproblem, while forming the main bottleneck to the method, is dwarfed by the initial Single approach. We argue that this is the case because a lot of columns are generated by Single, which we use to warm-start our MOO approaches.

IV. Modified KEP example and fairness schemes

To better understand our fairness scheme models, we apply them to the example of Figure 1. To this end, we add to the example a set of prioritized pairs V_H , as required by Aristotle's equity principle: over the rest of this section, we will to Figure IV.1. Based on Figure IV.1, we can compute the set $\mathcal{F}_G$ of feasible solutions, whose size is $|\mathcal{F}_G| = 4$. As stated in Section 4, we can only consider maximal exchanges: the chain (v_7, v_6, v_5) contains the only other chain as a subchain and therefore is always be part of maximal exchanges. Each cycle's selection is also exclusive to the other's; we can therefore focus on the choice between each cycle in the final distribution. We will denote the two maximal exchange plans as S_1 and S_2 , corresponding to cycle (v_1, v_2, v_3) (with chain (v_7, v_6, v_5)) and cycle (v_1, v_4) (with chain (v_7, v_6, v_5)) respectively.

Rawlsian justice seeks to maximize the minimum probability of selection over vertices. This value is maximized by setting $\delta(S_1) = \delta(S_2) = \frac{1}{2}$, where $\delta(S_1)$ is the probability of selecting cycle (v_1, v_2, v_3) and $\delta(S_2)$ is the probability of selecting cycle (v_1, v_4) . In Figure IV.1, cycle (v_1, v_2, v_3) contains 2 prioritized vertices, namely v_1 and v_3 . If the non-disjoint cycle (v_1, v_4) is selected instead, we have only v_1 . For Aristotle's equity principle, the first cycle will be selected as it contains higher priority pairs. For Nash's standard of comparison, the patient-donor pair selection probability could be used as an individual utility measure for each pair. Accordingly, the selection of cycle (v_1, v_2, v_3) is prioritized over cycle (v_1, v_4) since the sum of the log-probabilities over the graph is $2 \log(\delta(S_1)) + 1 \log(1 - \delta(S_1))$

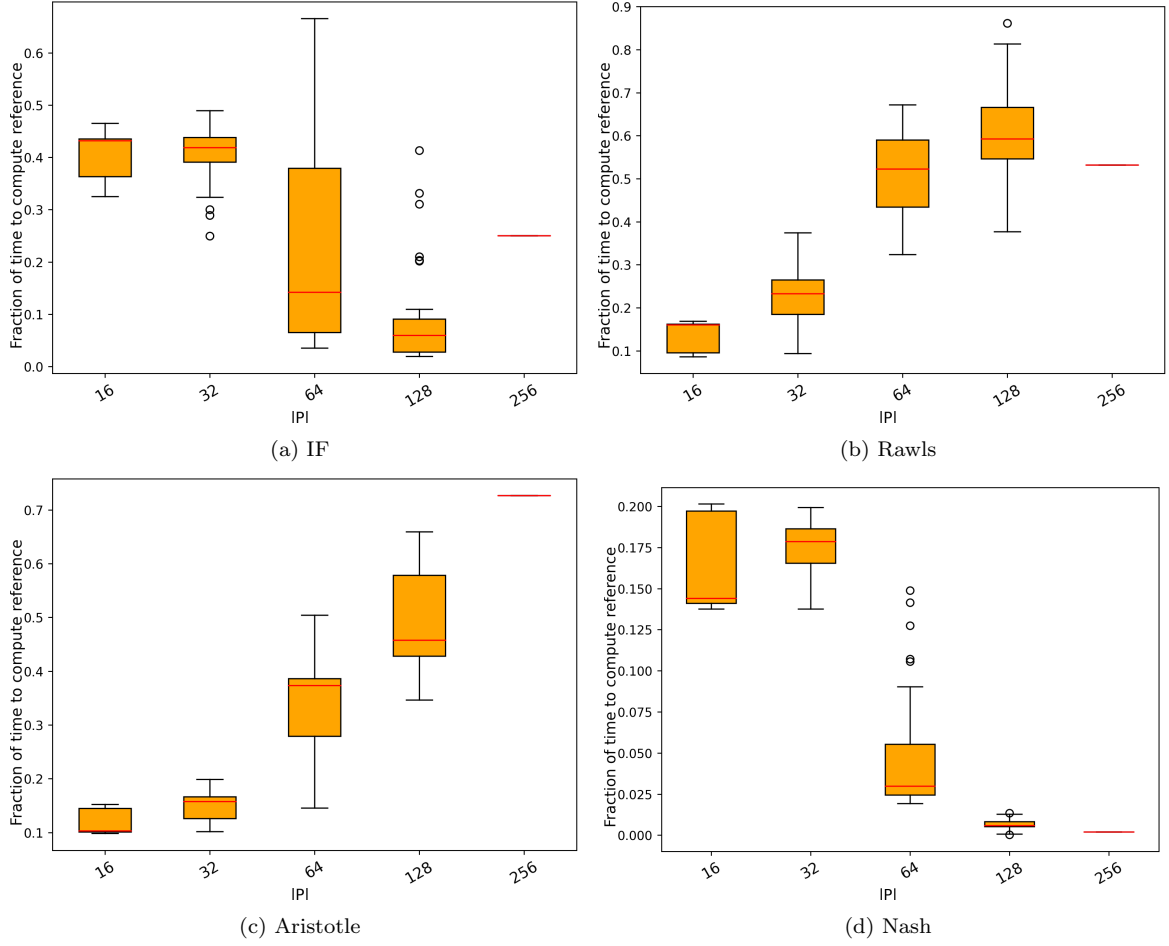


Figure III.2: Time spent computing the reference point

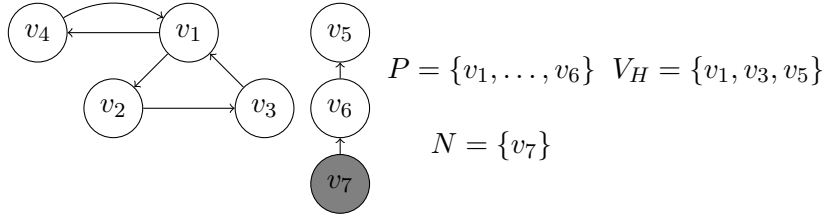


Figure IV.1: Example of a KEP graph

is maximized for $\delta(S_1) = \frac{2}{3}$, where $\delta(S_1)$ is defined as above. This can be seen by taking the derivative with respect to $\delta(S_1)$, setting it equal to 0 and solving the resulting equation. Changing the KEP objective to individual fairness leads to an average probability $z = \frac{2+\delta(S_1)}{4}$, where, again, $\delta(S_1)$ is the probability of selecting cycle (v_1, v_2, v_3) . With $\delta(S_1) = \frac{2}{3}$, the L_1 score is equal to $\frac{2}{3}$. If we increase the probability $\delta(S_1)$ to $\delta'(S_1) = \frac{2}{3} + \epsilon$ for a small $\epsilon > 0$, the score increases by $\frac{5\epsilon}{2}$. Similarly, for $\delta'(S_1) = \frac{2}{3} - \epsilon$, the score increases by $\frac{\epsilon}{2}$. The optimum is therefore found at $\delta(S_1) = \frac{2}{3}$ by convexity of the L_1 loss.

V. Other formulations of the NSWP

In this section of the appendix, we provide the missing dual formulations to the other primal problems introduced in the main body of the article.

V.I. NSWP with Rawlsian objective

$$\begin{aligned}
& \min -\alpha_0 + \alpha_1 d_1 + \alpha_2 d_2 && (\text{D}_{\text{Rawls}}) \\
& \text{s.t.} \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 1 \end{pmatrix} + u = 0 \\
& \quad \beta_v + \gamma_v = 0 && \forall v \in P' \\
& \quad -\alpha_2 - \sum_{v \in P'} \gamma_v = 0 \\
& \quad \alpha_0 - \sum_{v \in V(S) \setminus N} (\alpha_1 + \beta_v) + \lambda_S = 0 && \forall S \in \mathcal{F}_G \\
& \quad u \in \mathcal{Q}_r^3 \\
& \quad \gamma_v \geq 0 && \forall v \in P' \\
& \quad \alpha \in \mathbb{R}^3, \beta \in \mathbb{R}^{|P'|}, \lambda \geq 0.
\end{aligned}$$

V.II. NSWP with Aristotelian objective

$$\begin{aligned}
& \min -\alpha_0 + \alpha_1 d_1 + \alpha_2 d_2 && (\text{D}_{\text{Aristotle}}) \\
& \text{s.t.} \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 1 \end{pmatrix} + u = 0 \\
& \quad \alpha_0 - \sum_{v \in V(S) \setminus N} \alpha_1 - \sum_{v \in V(S) \cap P_H} \alpha_2 + \lambda_S = 0 && \forall S \in \mathcal{F}_G \\
& \quad u \in \mathcal{Q}_r^3 \\
& \quad \alpha \in \mathbb{R}^3, \lambda \geq 0.
\end{aligned}$$

V.III. NSWP with Nash principle of fairness

$$\begin{aligned}
& \min -\alpha_0 + \alpha_1 d_1 + \alpha_2 d_2 + \sum_{v \in P'} (w_v)_2 & (\text{D}_{\text{Nash}}) \\
& \text{s.t.} \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 1 \end{pmatrix} + u = 0 \\
& \quad \begin{pmatrix} \beta_v \\ \eta \end{pmatrix} + \begin{pmatrix} (w_v)_1 \\ (w_v)_3 \end{pmatrix} = 0 & \forall v \in P' \\
& \quad -\alpha_2 - \eta = 0 \\
& \quad \alpha_0 - \sum_{v \in V(S) \setminus N} (\alpha_1 + \beta_v) + \lambda_S = 0 & \forall S \in \mathcal{F}_G \\
& \quad u \in \mathcal{Q}_r^3 \\
& \quad w_v \in (K_{\text{exp}})^* & \forall v \in P' \\
& \quad \alpha \in \mathbb{R}^4, \beta \in \mathbb{R}^{|P'|}, \eta \in \mathbb{R}, \lambda \geq 0.
\end{aligned}$$

NSWP with individual fairness

$$\begin{aligned}
& \min -\alpha_0 + \alpha_1 d_1 + \alpha_2 d_2 & (\text{D}_{L_1}) \\
& \text{s.t.} \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 1 \end{pmatrix} + u = 0 \\
& \quad \begin{pmatrix} \eta \\ \beta_v \end{pmatrix} + w_v = 0 & \forall v \in P' \\
& \quad \alpha_2 - \eta = 0 \\
& \quad \sum_{v \in P'} \beta_v + |P'| \alpha_3 = 0 \\
& \quad \alpha_0 - \sum_{v \in V(S) \setminus N} (\alpha_1 + \alpha_3 + \beta_v) + \lambda_S = 0 & \forall S \in \mathcal{F}_G \\
& \quad u \in \mathcal{Q}_r^3 \\
& \quad w_v \in \mathcal{Q}^2 & \forall v \in P' \\
& \quad \alpha \in \mathbb{R}^4, \beta \in \mathbb{R}^{|P'|}, \eta \in \mathbb{R}, \lambda \geq 0.
\end{aligned}$$

VI. NSWP and various fairness objectives

Given the similarities of NSWP and SWP discussed in Section 8, we select the NSWP to evaluate this multi-objective framework across the different fairness objective functions.

RQ7: How does the NSWP fare with respect to each fairness objective? Again, we consider IF, Rawlsian justice, Aristotle and Nash's fairness principles.

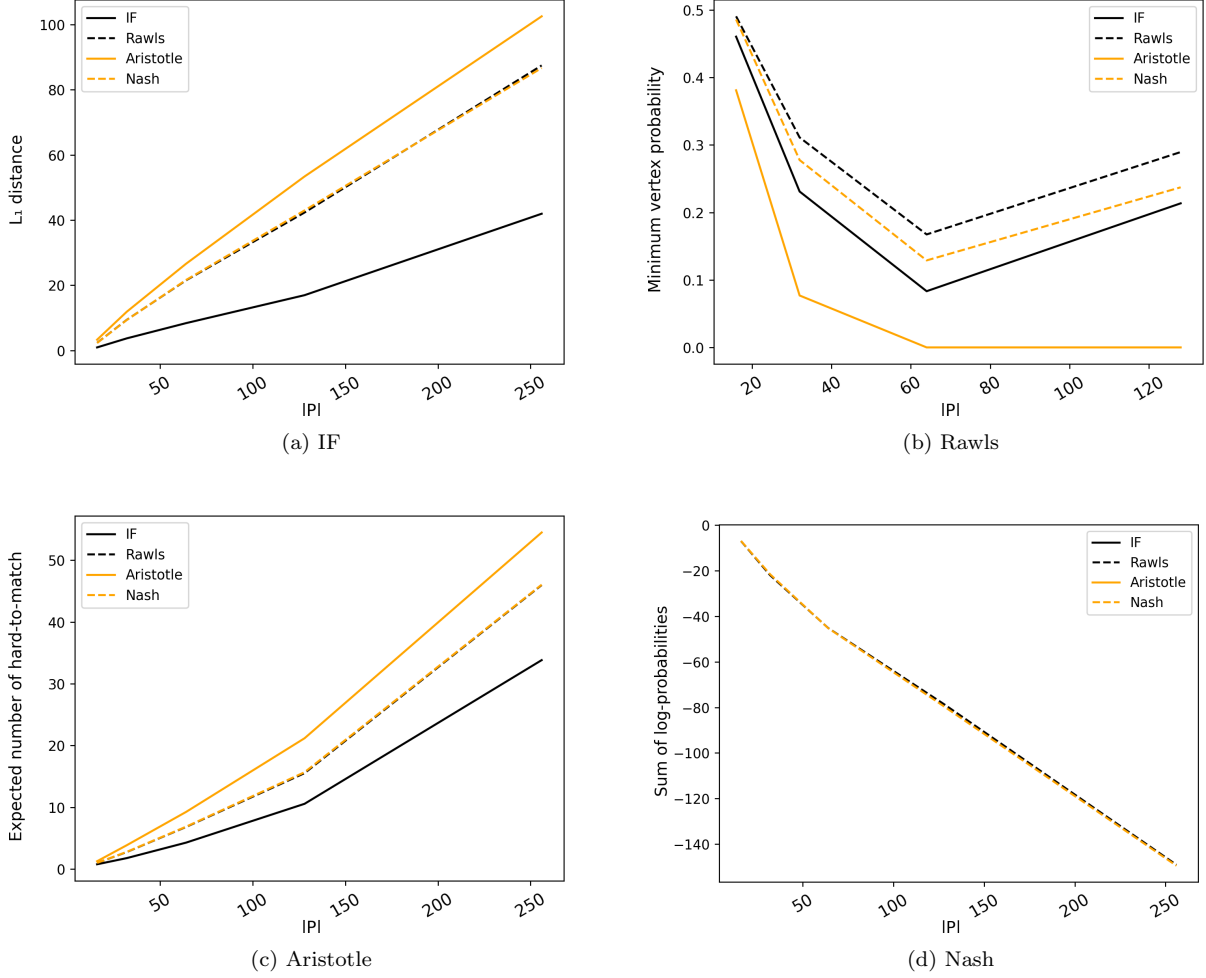


Figure VI.1: Comparison of the NSWP fairness schemes with respect to the values of each fairness objective

We formed Figure VI.1 by selecting the lotteries obtained by the NSWP for IF, Rawls, Aristotle and Nash, and evaluating them across different fairness scores (*i.e.*, the corresponding fairness scores of IF, Rawls, Aristotle and Nash). We can observe that the (NSWP) Rawlsian and Nash fairness schemes perform very similarly. This is in sharp contrast to IF and Aristotle’s fairness principle as they usually provide very different results. Aristotle’s principle is noteworthy as it performs the worst on all objectives, except when it corresponds to the one being optimized.

VII. Issue related to IF

The fairness scheme IF is problematic when we consider sub-optimal exchange plans in terms of the total number of transplants. This is because IF tries to minimize the distance to the average selection probability over all vertices in P' . However, if no vertex is selected, all probabilities are 0 and the “best” solution is to select the empty exchange plan. Consider an example over a graph with a single cycle of length K . Half of the time, we select the empty exchange plan, while we select the optimal exchange plan for the utilitarian objective the other half of the time (*i.e.*, the single cycle). This leads to very poor results in practice and is due to the fact that IF is poorly designed for the inclusion of suboptimal exchange plans. We note that this example is provided to highlight a particular failure mode of IF in a multi-objective setting and that there might exist other scenarios for

which performance can suffer. However, since the empty exchange plan is optimal with respect to this fairness objective, we expect this issue to be fairly recurrent, especially when there are few optimal exchange plans that maximize the number of transplants and that such exchange plans include most vertices in the KEP graph. Allocating a non-zero probability to the empty exchange plan in this scenario would proportionally be more detrimental to the expected number of transplants than in less degenerate cases. To resolve the issue, we propose an interesting fix to this issue that involves bilevel programming.

Definition 13. *A feasible exchange plan $F \in \mathcal{F}_G$ is maximal if it is not contained in some strictly larger exchange plan F' . In other words, there does not exist an $F' \in \mathcal{F}_G$ such that F is a subgraph of F' .*

Using this definition, we get rid of the optimality of the empty exchange plan for the fairness objective. However, because we add the constraint for maximal exchange plans, we must ensure that the generated solutions satisfy it. This leads to a bilevel formulation for the subproblem in the solution generation process.

$$\begin{aligned}
& \min \alpha_0 - \sum_{v \in V(F) \cap P} (\alpha_1 + \alpha_3 + \beta_v) & (\text{SP}) \\
& \text{s.t. } f_1(F') \leq f_1(F) \\
& \quad F' \in \arg \max \{f_1(S) : S \in \mathcal{F}_G \text{ s.t. } F \text{ is a subgraph of } S\} \\
& \quad F \in \mathcal{F}_G
\end{aligned}$$

The first constraint in (SP) ensures that we do not consider exchange plans that are not maximal and it will get rid of the aforementioned issue related to IF. It is an interesting research direction to explore how efficient this method can be made in practice and whether other improvements can be made to it. Before concluding our discussion on this topic, we call attention to the possibility of eliminating the empty exchange plan from consideration by adding a constraint to the multi-objective problem. However, if we consider a near-empty exchange plan, e.g., with a single cycle of length 2, then this exchange plan could be selected in the support of the optimal distribution because its f_2 score is close to 0 (for large $|P'|$). Adding a constraint that remove exchange plans with a small number of transplants (*i.e.*, below a threshold) is simple to implement experimentally, while it can address the issue with individual fairness in practice. Theoretically, the only way to eliminate any non-maximal exchange plan from consideration, however, is with the bilevel approach described above. It is currently unclear how practical this approach can be made and whether there exists degenerate instances where an added constraint would fail to either provide good fairness or utility scores.

VIII. Instances solved

In this section, we provide the number of instances solved by our methods. These numbers can be found in Tables VIII.1, VIII.2 and VIII.3.

IX. Practical recommendations

Based on the detailed information we provided on fairness schemes in this work, we condense the ideas that were discussed to provide practical guidelines that decision-makers and stakeholders can

$ P $	Percentage of NDDs			
	0%	5%	10%	15 %
16	10	0	10	10
32	10	10	10	10
64	10	10	10	10
128	10	10	10	10
256	6	8	4	0

(a) IF

$ P $	Percentage of NDDs			
	0%	5%	10%	15 %
16	10	0	10	10
32	10	10	10	10
64	10	10	10	10
128	10	10	10	10
256	6	8	4	0

(c) Aristotle

$ P $	Percentage of NDDs			
	0%	5%	10%	15 %
16	10	0	10	10
32	10	10	10	10
64	10	10	10	10
128	10	10	10	10
256	6	8	4	0

(b) Rawls

$ P $	Percentage of NDDs			
	0%	5%	10%	15 %
16	10	0	10	10
32	10	10	10	10
64	10	10	10	10
128	10	10	10	10
256	6	8	4	0

(d) Nash

Table VIII.1: Number of instances solved for each combination of $|P|$ and percentage of NDDs (for each fairness scheme) for a single (fairness) objective.

$ P $	Percentage of NDDs			
	0%	5%	10%	15 %
16	9	0	10	10
32	10	10	10	10
64	10	10	10	10
128	10	9	10	10
256	5	1	1	0

(a) IF

$ P $	Percentage of NDDs			
	0%	5%	10%	15 %
16	9	0	10	10
32	10	10	10	10
64	10	10	10	10
128	10	10	10	10
256	5	1	1	0

(c) Aristotle

$ P $	Percentage of NDDs			
	0%	5%	10%	15 %
16	9	0	10	10
32	10	10	10	10
64	10	10	10	10
128	10	10	10	10
256	5	1	1	0

(b) Rawls

$ P $	Percentage of NDDs			
	0%	5%	10%	15 %
16	9	0	10	10
32	10	10	10	10
64	10	10	10	10
128	10	10	10	10
256	5	1	1	0

(d) Nash

Table VIII.2: Number of instances solved for each combination of $|P|$ and percentage of NDDs (for each fairness scheme) by the SWP

$ P $	Percentage of NDDs			
	0%	5%	10%	15 %
16	9	0	10	10
32	10	10	10	10
64	10	10	10	10
128	10	10	10	10
256	4	1	1	0

(a) IF

$ P $	Percentage of NDDs			
	0%	5%	10%	15 %
16	9	0	10	10
32	10	10	10	10
64	10	10	10	10
128	10	10	10	10
256	4	1	1	0

(c) Aristotle

$ P $	Percentage of NDDs			
	0%	5%	10%	15 %
16	9	0	10	10
32	10	10	10	10
64	10	10	10	10
128	10	10	10	10
256	4	1	1	0

(b) Rawls

$ P $	Percentage of NDDs			
	0%	5%	10%	15 %
16	9	0	10	10
32	10	10	10	10
64	10	10	10	10
128	10	10	10	10
256	4	1	1	0

(d) Nash

Table VIII.3: Number of instances solved for each combination of $|P|$ and percentage of NDDs (for each fairness scheme) by the NSWP

follow when implementing fairness schemes for KEPs.

- The SWP is more efficient to solve than the NSWP based on the availability of highly efficient mixed-integer linear programming software. In our experiments, we computed the nadir point for the SWP to ensure that both objectives were scale-comparable, but this can be achieved in an ad-hoc manner rather efficiently by fine-tuning the weights λ .
- Fairness schemes run on their one can be risky since they can imply a large loss for the utilitarian objective. By combining them (through the SWP or the NSWP) with the utilitarian objective

(Section 6), or by introducing prerequisites (constraints) on the utilitarian value (Section 7.2) prevents such undesirable solutions. More specifically, this modifies the default fairness scheme into a more robust alternative.

- NSWP-Nash and NSWP-Rawls are similar and for this reason, we recommend using NSWP-Rawls because of its superior computational performance and better POF. NSWP-Aristotle should be used to understand the effect of protected group groups of patients.
- NSWP-IF currently has some limitations when it is not restricted to maximal exchange plans. Again, the introduction of an additional constraint to prevent the inclusion of empty or near-empty exchange plans can serve to counter this issue. It is important to note, however, that this can potentially impact the performance of this fairness scheme, such as the POF.