TWO-PLAYER KIDNEY EXCHANGE

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Abstract

Kidney exchange programs have been set in several countries within national, regional or hospital frameworks, to increase the possibility of kidney patients being transplanted. For the case of hospital programs, it has been claimed that hospitals would benefit if they collaborated with each other, sharing their internal pools and allowing transplants involving patients from different hospitals. However, it has been observed that each hospital is a self-interested agent that aims to maximize the number of its patients receiving a kidney. Therefore, the design of the exchange market must comply with each hospital's objective and it is crucial to get a game outcome that maximizes the social welfare, i.e., the maximum number of exchanges.

Players

Player A controls the incompatible patient-donor nodes

Player B controls the incompatible patient-donor nodes

Edges between patient-donor nodes represent a compatible exchange.

The players feasible strategies are matchings among their nodes, i.e., a set of internal edges such that no two edges share a common node.

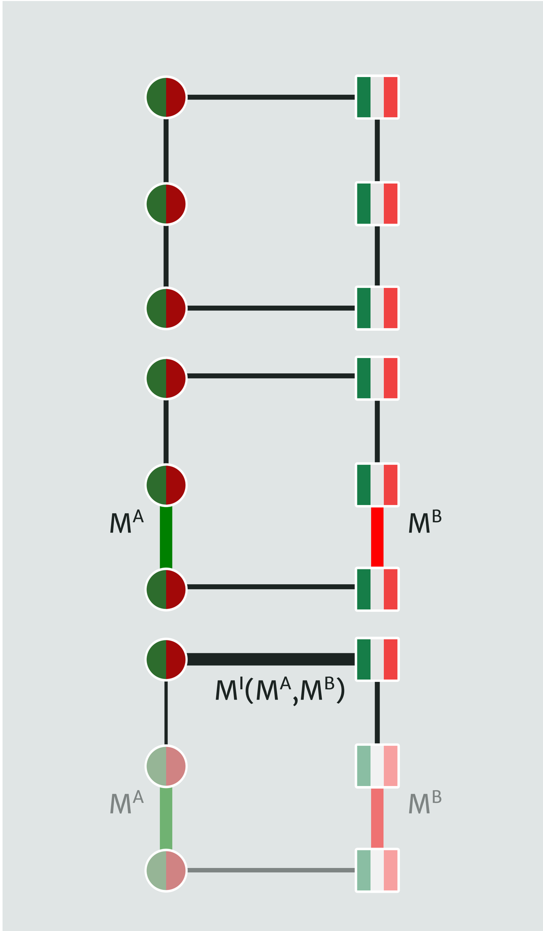
Game instructions

Simultaneously, player A and B reveal their incompatible pairs and the graph game is drawn.

Simultaneously, player A and B present their internal exchanges (matchings), M^A and M^B, respectively.

The system computes the external exchanges, $M^I(M^A,M^B)$, among unmatched nodes such that the number of transplants is maximized.

In the example: Player A utility is 3 and Player B utility is 3.

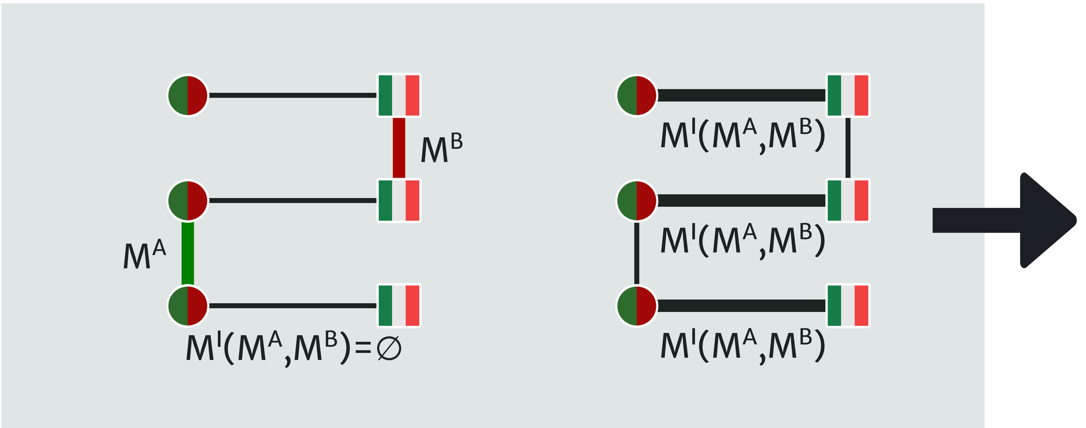


Game expected outcome: NASH EQUILIBRIUM

A player A's matching M^A and a player B's matching M^B is a **Nash Equilibrium** if

$$2|M^A| + |M^I(M^A,M^B)| \geq 2|R^A| + |M^I(R^A,M^B)| \quad \forall \text{ matching } R^A \text{ of } G^A$$

$$2|M^B| + |M^I(M^A,M^B)| \geq 2|R^B| + |M^I(M^A,R^B)| \quad \forall \text{ matching } R^B \text{ of } G^B$$



There are instances with more than one Nash equilibrium.

Refinement of the Nash equilibrium: **social welfare equilibrium (SWE).**

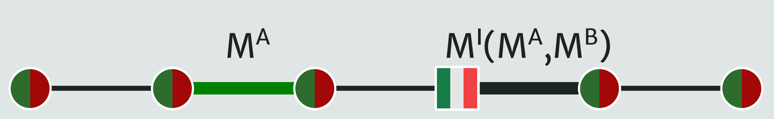
A **social welfare equilibrium** is a Nash equilibrium that is also a social optimum, i.e., it is a Nash equilibrium such that the maximum number of exchanges is achieved.

A SOCIAL WELFARE EQUILIBRIUM IS THE RATIONAL STRATEGY

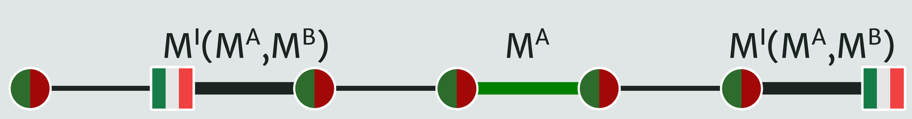
Complete characterization of optimal strategies

Player A is playing an optimal strategy iff there is no path such that:

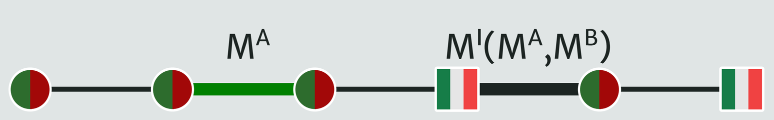
Case i. Player A can increase her utility by 2 units.



Case ii. Player A can increase her utility by 1 unit.



Case iii. Player A can increase her utility by 1 unit.



The symmetric result for player B also holds.

Any Nash equilibrium is dominated by a social welfare equilibrium:

```
INPUT: Instance G, a NE M of G
OUTPUT: M if it is a SWE, else a SWE dominating it
1.  S ← a maximum matching of G
2.  IF |M| = |S|:
3.      RETURN M
4.  END IF
5.  t ← 1
6.  P^t ← paths from M ⊕ S with both extreme edges in S # M-augmenting paths
7.  M^t ← M ⊕ p_1 ⊕ p_2 ⊕ ... ⊕ p_r where {p_1, p_2, ..., p_r} = P^t
8.  I ← {e: e ∈ E^I ∩ M^t}
9.  WHILE I  ≠ ∅
10.    select an edge (v_0, v_1) ∈ I # assume v_0 ∈ V^B  and v_1 ∈ V^A
11.    x ← M^t-alternating path of type ii. ∈ G^A(M^t ∩ E^B) starting in (v_0, v_1)
12.    WHILE path x = (v_0, v_1, ..., v_2m) is found
13.          j ← max_{i=0,...,2m-1} {i : (v_i, v_i+1) ∈ q for some q ∈ P^t}
14.          y ← (u_0, u_1, ..., u_k, u_k+1, ..., u_f) ∈ P^t used to determine j
                 with (u_k, u_k+1) = (v_j, v_j+1)
15.          z ← (v_2m, v_2m-1, ..., v_j+1, u_k+2, ..., u_f)
16.          M^t+1 ← M^t ⊕ y ⊕ z
17.          P^t+1 ← (P^t - {y}) ∪ {z}
18.          t ← t+1
19.          I ← {e: e ∈ E^I ∩ M^t}
20.          G' ← subgraph of G^A(M^t ∩ E^B) induced by considering only edges
                  of x from v_0 to v_j = u_k and of y from u_0 to u_k = v_j
21.          x ← M^t-alternating path of type ii. in G' starting in (v_0, v_1)
22.    END WHILE
23.    repeat steps 10 to 21 inverting the roles of players A and B
24.    I = I - {(v_0,v_1)}
25. END WHILE
26. RETURN M^t.
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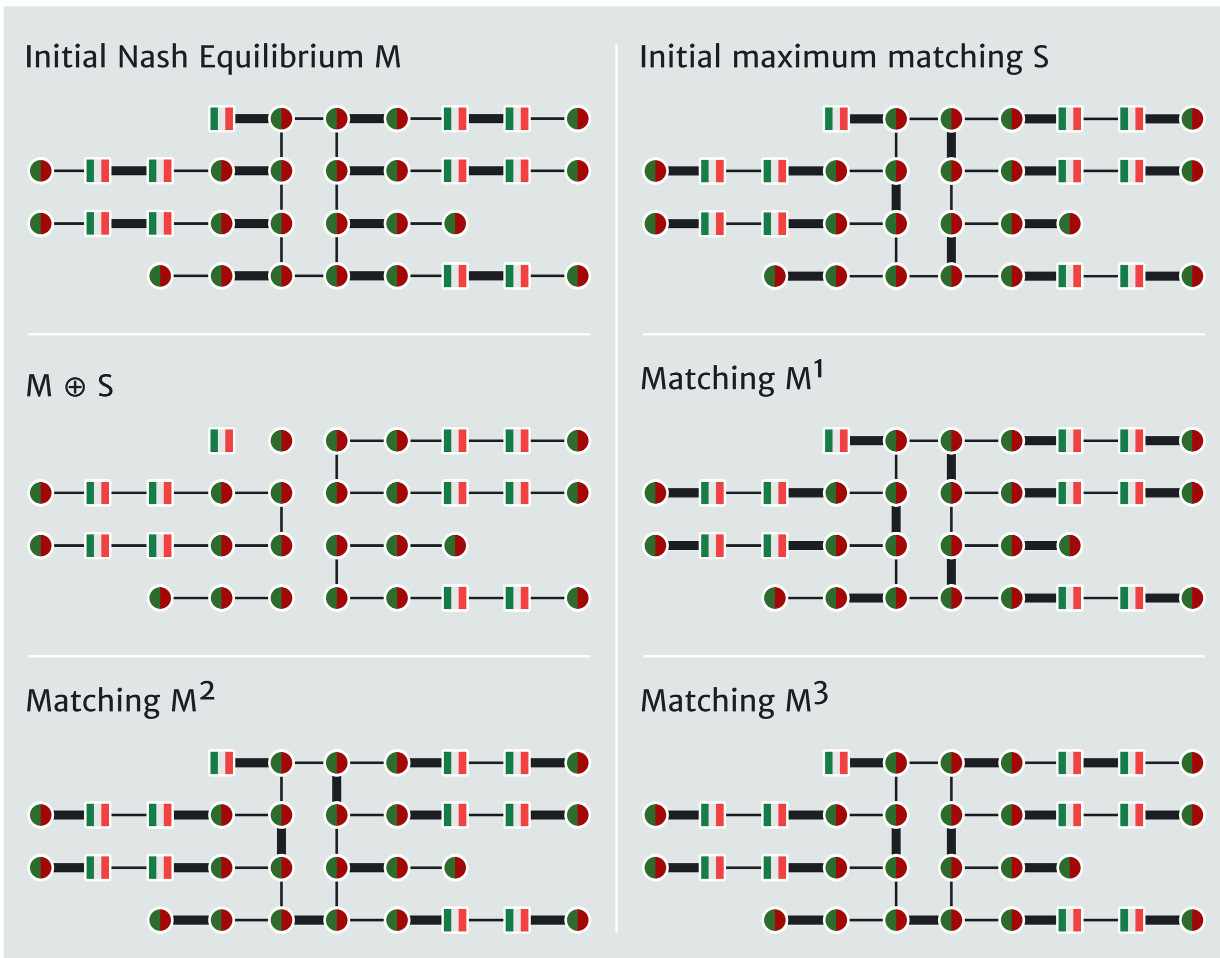
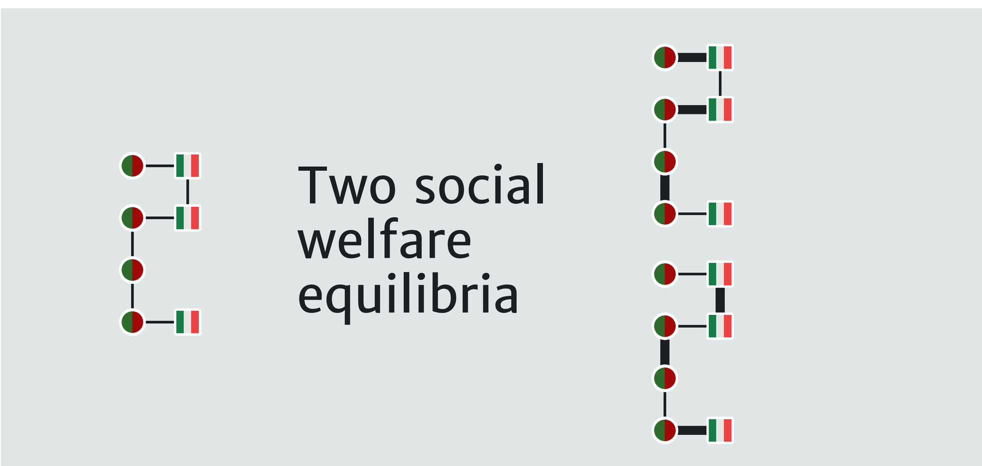


The social welfare equilibrium is not enough to guarantee uniqueness:



Refinement of the social welfare equilibrium

```
INPUT: Instance G
OUTPUT: a SWE that minimizes the number of external exchanges
FOR e ∈ E^A ∪ E^B
     w_e ← 2 + 2|V|
END FOR
FOR e ∈ E^I
     w_e ← 1 + 2|V|
END FOR
M ← maximum weighted matching in G given edge weights w_e
RETURN M.
```

THE ALGORITHM:

- runs in polynomial time
- outputs a social optimum
- outputs a Nash equilibrium
- minimizes the number of external exchanges among the set of social welfare equilibria

THEOREM:

There is always a social welfare equilibrium. Moreover, the one minimizing the number of external exchanges is unique in terms of the players utilities.

Acknowledgments

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